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A THEORY OF PHYSICAL VACUUM

A New Paradigm

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The monograph contains the results of the programe of the geometrization of the equations of physics advanced by Einstein at the beginning of the 20^{th} century and developed by the author who gives the idea of the universal principle of relativity and the theory of physical vacuum and analyses theoretical and experimental effects of the theory.

The second part of the monograph develops the mathematical system of the physical vacuum theory. It contains the main characteristics of the absolute parallelism geometry in vector and spinor basis.

The book is intended for specialists in theoretical physics, teachers, post-graduates, students and everyone interested in new physical theories.

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Preface

This book gives a concise presentation of ideas and methods used by the author to develop the Clifford-Einstein program of the geometrization of the equations of physics, and also to solve various fundamental problems of modern theoretical physics proceeding from the concept of the universal principle of relativity and the theory of physical vacuum. In his studies the author made an attempt to combine phenomena of seemingly different nature and to sketch a coherent picture of modern physics.

The author is most grateful to V. Yu. Tatur and all those who, directly or indirectly, made the publication of this book possible. Special thanks are due to my friends and colleagues E. A. Gubarev, A. N. Sidorov, and I. A. Volodin.

Many ideas expounded in this book were presented in my first monograph published in 1979 with a support of M. A. Adamenko and I. S. Lakoba at Moscow University Press.

I remember with gratitude my productive talks with V. Skalsky, an Associate Professor at Slovak Polytechnic, who made some valuable points about various vacuum states of matter.

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The attention and support of all these persons contributed enormously to the publication of this book.

Last but not least, the author must record his deep obligation to Elena Turantaeva who was good enough to edit the book and read the proofs.

1993

Gennady Shipov

Preface to English edition

The translation of the book into English is shortened as far as it doesn't contain the 5^{th} chapter of the Russian edition. This chapter is dedicated to phenomena of seemingly different nature and its absence doesn't influence the main scientific results.

1998

Gennady Shipov

Conventions[1]

Three-dimensional tensor indices are denoted by the Greek letters $\alpha, \beta, \gamma, \ldots$ and take the values 1, 2, 3.

Three-dimensional vectors (e.g., linear and angular velocity) are denoted as: $\vec{v}$ and $\vec{\omega}$ or $\mathbf{v}$ and ω.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k \ldots$; they assume the values 0, 1, 2, 3. Letters from the first part of the alphabet $(a, b, \ldots, h)$ are used as tetrad indices, e.g., $e^i{}_a$, $\quad (a = 0, 1, 2, 3)$.

Spinor indices in the spinor Δ-basis are denoted by Roman capitals $A, B \ldots, \dot{C} \ldots \dot{D}$ and take the values 0,1 or $\dot{0}, \dot{1}$. Spinor indices in the Γ-basis are labeled by Greek letters $\alpha, \beta, \ldots, \dot{\gamma}, \dot{\delta} \ldots$.

Symmetrization and antisymmetrization of pairs of indices:

$$S_{(ij)} = \frac{1}{2}(S_{ij} + S_{ji}), \quad S_{[ij]} = \frac{1}{2}(S_{ij} - S_{ji}).$$

Exclusion of an index from symmetrization or antisymmetrization:

$$S_{(i|j|k)} = \frac{1}{2}(S_{ijk} + S_{kji}), \quad S_{[i|j|k]} = \frac{1}{2}(S_{ijk} - S_{kji}).$$

Passing over to local (tetrad) indices: $S^a{}_{bc} = e^a{}_i S^i{}_{jk} e^j{}_b e^k c$.

External product: $e^a \wedge e^c = a^a e^c - e^c e^a$.

The Levi-Chivita pseudotensor: ε_{ijkm}; dual tensor: $\overset{*}{S}_{ij} = \frac{1}{2}\varepsilon_{ijkm} S^{km}$.

The matrix representation of

(a) tensor quantities :

$S^a{}_{bk}$ or, discarding the matrix indices a and b, $S^a{}_{bk} \to S_k$;

(b) spin-tensor quantities: $S^{A\dot{B}}{}_{C\dot{D}k} \to S_k$.

A matrix product: $[T_m, T_k] = T_m T_k - T_k T_m$.

Hermitian conjugate matrices: $S^+{}_{B\dot{D}kn}$.

Derivatives

Partial derivatives with respect to the translational coordinates x^i are labeled by a comma in front of an index, i.e., $f_{,k} = \partial f / \partial x^k = \partial_k f$; a covariant derivative with respect to the Christoffel symbols $\Gamma^i{}_{jk}$ is denoted by ∇_k or $\nabla_k u^i = \partial_k u^i + \Gamma^i{}_{jk} u^j$.

A local covariant derivative: $\nabla_a u^b = \partial_a u^b + \Gamma^b{}_{ca} u^c$.

A covariant derivative $\overset{*}{\nabla}_k$ with respect to the connection $\Delta^i_{jk} = e^i_a e^a_{j,k}$ of the A_4 geometry: $\overset{*}{\nabla}_k \, u^i = \partial_k u^i + \Delta^i_{jk} u^j$.

[1] The following is a list of only some important notations. All the conventions are explained in the text.

An external derivative: d.
A spinor derivative: $\partial_{A\dot{B}}$.

Translational metric and tetrads

Translational coordinates: x^0, x^1, x^2, x^3.
The metric signature: $(+---)$.
A translational linear element:
$ds^2 = \eta_{ab} e^a{}_i e^b{}_j dx^i dx^j$, $\eta_{ab} = \eta^{ab} = \mathrm{diag}(1 -1 -1 -1)$
The structural equations of the group of translations of the A_4 geometry:
$\nabla_{[a}\nabla_{b]} x^i = -\Omega_{ab}{}^{c}\nabla_c x^i$.
1-form of the tetrad: $e^a = e^a{}_i dx^i$.

Rotational metric and torsion

Rotational coordinates: $\varphi_1, \varphi_2, \varphi_3, \theta_1, \theta_2, \theta_3$.
Rotational metric: $d\tau^2 = d\chi^b{}_a d\chi^a{}_b = T^a{}_{bi} T^b{}_{aj} dx^i dx^j$,
$d\chi_{ab} = -d\chi_{ab}$.
The torsion of A_4 geometry: $\Omega_{jk}{}^{i} = e^i{}_a e^a{}_{[k,j]} = \frac{1}{2} e^i{}_a (e^a{}_{k,j} - e^a{}_{j,k})$.
The contorsion tensor of A_4 geometry (the rotational Ricci coefficients):

$$T^i_{jk} = -\Omega_{jk}{}^{i} + g^{im}(g_{js}\Omega_{mk}{}^{s} + g_{ks}\Omega_{mj}{}^{s}) = e^i{}_a \nabla_k e^a{}_j.$$

1-form of contorsion: $T^a{}_b = T^a{}_{bk} dx^k = T^a{}_{bc} e^c$, $T_{(ab)} = 0$.
The structural equations of a rotational group (the matrix indices are discarded): $\nabla_{[k}\nabla_{m]} e^i = \frac{1}{2} R_{km} e^i$, where $R_{km} = 2\nabla_{[m} T_{k]} + [T_m, T_k]$.

Connection and curvature of A_4 geometry

Connection: $\Delta^i{}_{jk} = \Gamma^i{}_{jk} + T^i{}_{jk} = e^i{}_a e^a{}_{j,k}$;
$\Delta^i_{[jk]} = T^i_{[jk]} = -\Omega_{jk}{}^{i}$; $\Delta^i_{(jk)} = \Gamma^i{}_{jk} + g^{im}(g_{js}\Omega_{mk}{}^{s} + g_{ks}\Omega_{mj}{}^{s})$.
Curvature:

$$\begin{aligned} S^i_{jkm} &= 2\Delta^i{}_{j[m,k]} + 2\Delta^i{}_{s[k}\Delta^s{}_{|j|m]} = \\ &= R^i_{jkm} + 2\nabla_{[k} T^i_{|j|m]} + 2T^i_{c[k} T^c_{|j|m]} = 0 \end{aligned}$$

where $R^i_{jkm} = 2\Gamma^i{}_{j[m,k]} + 2\Gamma^i{}_{s[k}\Gamma^s{}_{|j|m]}$ — the Riemann tensor.
1-form of connection: $\Delta^a{}_b = \Delta^a{}_{bk} dx^k = \Delta^a{}_{bc} e^c$.
The Cartan structural equations:
(a) first structural equations: $de^a - e^c \wedge T^a{}_c = 0$;
(b) second structural equations: $R^a{}_b + dT^a{}_b + T^c{}_b \wedge T^a{}_c = 0$.

Spinor Δ-basis

Newman-Penrose symbols: $\sigma_i^{A\dot{B}}$.

Translational metric: $g_{ij} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\sigma_i^{A\dot{B}}\sigma_j^{C\dot{D}}$, where ε^{AB} is a fundamental spinor

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon^{\dot{C}\dot{D}} = \varepsilon_{\dot{C}\dot{D}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The rotational Ricci coefficients:

$$T_{A\dot{B}C\dot{D}k} = \sigma^i_{C\dot{D}}\nabla_k\sigma_{A\dot{B}i}.$$

The rotational Ricci coefficients in terms of Carmeli matrices: $T_{A\dot{B}}$ with matrix elements $(T_{A\dot{B}})_C{}^D$.

The Riemann curvature in terms of Carmeli matrices: $R_{A\dot{B}C\dot{D}}$.

The equations of physical vacuum written in Carmeli matrices:

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - \\ -(T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}, \qquad (A^s)$$

$$R_{F\dot{E}D\dot{B}} = \partial_{D\dot{B}}T_{F\dot{E}} - \partial_{F\dot{E}}T_{D\dot{B}} - (T_{D\dot{B}})_F{}^S T_{S\dot{E}} - (T^+_{\dot{E}D})^{\dot{F}}{}_{\dot{B}}T_{F\dot{F}} + \\ +(T_{F\dot{E}})_D{}^S T_{S\dot{B}} + (T^+_{\dot{B}F})^{\dot{F}}{}_{\dot{E}}T_{D\dot{F}} + [T_{F\dot{E}}, T_{D\dot{B}}]. \qquad (B^{s+})$$

+ Hermitially conjugate equations.

Part II

GEOMETRY OF ABSOLUTE PARALLELISM

Introduction

Geometry with absolute parallelism was first considered in 1923-24 in the works of Weitzenbock [1, 2] and Vitali [3, 4]. Weitzenbock suggested that there exist in the n-dimensional manifold M with coordinates $x^1, \ldots, x^n$ of Riemannian spaces with a zero Riemann-Christoffel tensor

$$S^i{}_{jkm} = 2\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]} = 0. \qquad (4.1)$$

Relationship (4.1) was regarded as the condition of parallel displacement of an arbitrary vector in a given space in the absolute (independent of path) sense. In 1924 Vitali introduced the concepts of the connection of absolute parallelism [3]

$$\Delta^k_{ij} = e^k{}_a e^a{}_{i,j}, \qquad (4.2)$$

$$,j = \tfrac{\partial}{\partial x^j}, \quad i, j, k \ldots = 0, 1, 2, 3,$$
$$a, b, c \ldots = 0, 1, 2, 3,$$

where $e^k{}_a$ and $e^a{}_i$ are basic vectors defined at each point of space and translatable in the absolute sense to any point of the space in any direction. Weitzenbock [5] showed that the connection (4.2) can be represented as the sum

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}, \qquad (4.3)$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \qquad (4.4)$$

are the Christoffel symbols and

$$T^i_{jk} = -\Omega_{jk}^{\cdot\cdot i} + g^{im}(g_{js}\Omega^s_{mk} + g_{ks}\Omega_{mj}^{\cdot\cdot s}) \qquad (4.5)$$

are the Ricci rotation coefficients [6] for the basis $e^a{}_i$.

The tensor $\Omega_{jk}^{\cdot\cdot i}$, defined as

$$\Omega_{jk}^{\cdot\cdot i} = e^i{}_a e^a_{[k,j]} = \frac{1}{2} e^i{}_a (e^a{}_{k,j} - e^a{}_{j,k}), \qquad (4.6)$$

came to be known as the anholonomity object [7], therefore the emergence of the geometry of absolute parallelism continued the development of anholonomic differential geometry [8].

Cartan and Schouten [9, 10], proceeding from the group properties of the space of constant curvature, introduced the connection (4.3), in which the components of the Ricci rotation coefficients (4.5) are constants.

Cartan and Schouten reasoned as follows. Suppose that in a n - dimensional differentiable manifold M with the coordinates $x^1, \ldots, x^n$ we have n contravariant vector fields

$$\xi^j_a = \xi^j_a(x^k), \qquad (4.7)$$

where

$$a, b, c \ldots = 1 \ldots n$$

are vector indices, and

$$i, j, k \ldots = 1 \ldots n$$

are coordinate indices.

Suppose that

$$\det(\xi^j_a) \neq 0$$

and that the functions ξ^j_a satisfy the equations

$$\xi^j_a \xi^k{}_{b,j} - \xi^i_b \xi^k{}_{a,j} = -C_{ab}^{\cdot\cdot f} \xi^k_f,$$

where the constants $C_{ab}^{\cdot\cdot f}$ have the following properties:

$$C_{ab}^{\cdot\cdot f} = -C_{ba}^{\cdot\cdot f}, \tag{4.8}$$

$$C_{fb}^{\cdot\cdot a} C_{cd}^{\cdot\cdot f} + C_{fc}^{\cdot\cdot a} C_{db}^{\cdot\cdot f} + C_{fd}^{\cdot\cdot a} C_{bc}^{\cdot\cdot f} = 0. \tag{4.9}$$

We can then say that we have an n-parametric simple transitive group (group T_n) operating in the manifold such that $C_{ab}^{\cdot\cdot f}$ are structural constants of the group that obey the Jacobi identity (4.9). The vector field ξ^j_b is said to be infinitesimal generators of the group.

Let now the basis e^k_b, defined at each point of the manifold M, meet the condition

$$\det(e^j{}_a) \neq 0.$$

If we suppose that

$$e^j{}_a(x^k_0) = \xi^j_a(x^k_0),$$

where x^k_0 are the coordinates of some arbitrary point P, then we have for the function $e^j{}_a(x^k_0)$ the equations

$$e^j{}_a e^k{}_{b,j} - e^j{}_b e^k{}_{a,j} = -C_{ab}^{\cdot\cdot f} e^k{}_f. \tag{4.10}$$

It follows from the normalization condition for the basis

$$e^a{}_i e^j{}_a = \delta^j_i, \quad e^a{}_i e^i{}_b = \delta^a_b, \tag{4.11}$$

and from (4.10), that

$$C_{jk}^{\cdot\cdot i} = 2e^i{}_a e^a{}_{[k,j]} = e^i{}_a C^{\cdot\cdot a}_{bc} e^b_j e^c_k. \tag{4.12}$$

Comparing (4.8) and (4.6), we see that

$$\Omega_{jk}^{\cdot\cdot i} = \frac{1}{2} C_{jk}^{\cdot\cdot i},$$

i.e., the components of the anholonomity object of a homogeneous space of absolute parallelism are constant.

It is easily seen that the connection (4.2) possesses a torsion. In our specific case

$$\Delta^k_{[ij]} = -\Omega_{ij}^{\cdot\cdot k} = T^k_{[ij]} = -\frac{1}{2} C_{jk}^{\cdot\cdot i}.$$

It was exactly in this manner that Cartan and Schouten introduced connection with torsion [9, 10]. Therefore, the development of the geometry of absolute parallelism brought about the emergence of the Riemann-Cartan geometry with the connection

$$\Delta_{ijk} = \Gamma_{ijk} + \frac{1}{2}(C_{ijk} - C_{jki} - C_{kij}), \tag{4.13}$$

where $S_{ijk} = -\frac{1}{2}C_{ijk}$ is the torsion of space.

Further development of the geometry of absolute parallelism in the n-dimensional differentiable manifold M with coordinates $x^1, \ldots, x^n$ (geometries A_n) is described in the works of Bortolotti [11–14], Griss [15], Schouten [16, 17], Eisenhart [18] and other authors [19-25]. Specifically Bortolotti [12] was the first to point out that the Cartan-Schouten connection and the Weinzbock-Vitali (4.2) connection is one and the same thing. Besides, Bortolotti showed that the tensor (4.1) can be represented as the sum

$$S^i{}_{jkm} = R^i{}_{jkm} + 2\nabla_{[k}T^i{}_{|j|m]} + 2T^i{}_{c[k}T^c{}_{|j|m]} = 0, \tag{4.14}$$

where

$$R^i{}_{jkm} = 2\Gamma^i_{j[m,k]} + 2\Gamma^i_{s[k}\Gamma^s_{|j|m]} \tag{4.15}$$

is the Riemann tensor, and the quantities T^i_{jk} are given by (4.5).

In 1937 Thomas [20, 21] approached absolute parallelism as parallel displacement of vectors "in toto," since the connection of space A_n (just as that of a flat space E_n) is integratble. Therefore, a vector specified at some point A_n can be specified at any other point of space. Lastly, the works [23–25] give a classification of spaces with absolute parallelism.

Geometry A_4 has been first used by Einstein [26] in applications to problems of theoretical physics. The scientist made an attempt to combine the equations of his theory with the equations of the Maxwell-Lorentz electrodynamics [27]. We note in passing that within the framework of the geometry of absolute parallelism Einstein has written most (all in all 13) works.

By developing Einstein's program to construct a unified field theory, this author came to the conclusion that it is necessary to use the A_4 geometry as a geometry of space of events in universal relativity theory and the theory of physical vacuum. Unlike Einstein and his following, the author employed Cartan's structural equations of the geometry of absolute parallelism, which are generalizations of Einstein's vacuum equations $R_{ik} = 0$ for the case where the energy-momentum tensor on the right-hand side of Einstein's equations is geometric in nature.

The program of unified field theory put forward by Einstein boils down to solving two strategic problems of modern theoretical physics:

(a) the minimum program has it as its goal to geometrize the equations of electromagnetic field and to combine them with the equations of Einstein's theory of gravitation;

(b) the maximum program is aimed at the search for completely geometrized equations of the gravitational and electromagnetic field (including sources), i.e., the geometrization of the fields that form matter.

Although much time was devoted to this search (around 30 years), Einstein failed to solve the problem in a form acceptable to science. Together with many outstanding scientists of the time he wrote a wealth of works relying on various geometries. But all of them failed to meet the above requirements (a) and (b). Also, it was unclear how to geometrize spin fields (e.g., Dirac's field) that are sources of electromagnetic fields. Wheeler added to the program of unified field theory a further point that required a spinor treatment of the equations of the unified field. The latter condition can be met in the case where the main geometric quantities of the theory are spinors rather than tensors. A spinor treatment of classical geometries was given in the works by Penrose [38, 40, 54], which was of much help to me in my constructing a theory of physical vacuum, a present-day outgrowth of Einstein's program of unified field theory.

Chapter 5

Geometry of absolute parallelism in vector basis

5.1 Object of anholonomicity. Connection of absolute parallelism

Consider a four-dimensional differentiable manifold with coordinates x^i ($i = 0,1,2,3$) such that at each point of the manifold we have a vector $e^a{}_i$ ($i = 0,1,2,3$) and a covector $e^j{}_b$ ($b = 0,1,2,3$) with the normalization conditions

$$e^a{}_i e^j{}_a = \delta^j_i, \quad e^a{}_i e^i{}_b = \delta^a_b. \tag{5.1}$$

For arbitrary coordinate transformations

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^k} dx^k \tag{5.2}$$

in coordinate index i the tetrad $e^a{}_i$ transforms as a vector

$$e^a{}_{i'} = \frac{\partial x^i}{\partial x^{i'}} e^a{}_i. \tag{5.3}$$

In the process, in the tetrad index a relative to the transformations (5.2) it behaves as a scalar.

Tetrad $e^a{}_i$ defines the metric tensor of a space of absolute parallelism

$$g_{ik} = \eta_{ab} e^a{}_i e^b_k, \eta_{ab} = \eta^{ab} = \mathrm{diag}(1\ -1\ -1\ -1) \tag{5.4}$$

and the Riemannian metric

$$ds^2 = g_{ik} dx^i dx^k. \tag{5.5}$$

Using the tensor (5.4) and the normal rule [29], we can construct the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}). \tag{5.6}$$

that transform following a nontensor law of transformation [29]

$$\Gamma^{k'}_{j'i'} = \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} \Gamma^k_{ji} \tag{5.7}$$

with respect to the coordinate transformations (5.2). In the relationship (5.6) and farther on we will denote the partial derivative with respect to the coordinates x^i as

$$,k = \frac{\partial}{\partial x_k}. \tag{5.8}$$

Differentiating the arbitrary vector $e^a{}_i$ gives

$$e^a{}_{i,j'} = \frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j}. \tag{5.9}$$

Applying the differentiation operation (5.9) to the relationship (5.3) gives

$$e^a_{i',j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} e^a{}_i. \tag{5.10}$$

Alternating the indices i' and j' and subtracting from (5.10) the resultant expression, we have

$$e^a{}_{i',j'} - e^a{}_{j',i'} = (e^a{}_{i,j} - e^a{}_{j,i}) \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}.$$

Considering (5.3), we can rewrite this relationship in the form

$$e^{k'}{}_a (e^a{}_{i',j'} - e^a{}_{j',i'}) = e^k{}_a (e^a{}_{i,j} - e^a{}_{j,i}) \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k}.$$

By definition, the differential

$$ds^a = e^a{}_i dx^i \tag{5.11}$$

is said to be complete, if the following relationship holds:

$$e^a{}_{i,j} - e^a{}_{j,i} = 0. \tag{5.12}$$

Otherwise, for $e^a{}_{i,j} - e^a{}_{j,i} \neq 0$, the differential (5.11) is not integrable (equality (5.12) is the condition of integration for the relationship (5.11)).

We will introduce the following geometric object [30]

$$\Omega_{jk}^{\cdot\cdot i} = e^i{}_a e^a_{[k,j]} = \frac{1}{2} e^i{}_a (e^a{}_{k,j} - e^a{}_{j,k}) \tag{5.13}$$

with a tensor law of transformation relative to the coordinate transformations (5.2)

$$\Omega_{j'k'}^{\cdot\cdot i'} = \Omega_{jk}^{\cdot\cdot i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^i}. \tag{5.14}$$

Clearly, if the condition (5.12) is met, this object vanishes. In that case, tetrad $e^a{}_i$ is holonomic and the metric (5.5) characterizes holonomic differential geometry. If the object (5.13) is nonzero, we deal with anholonomic differential geometry, and the object (5.13) itself is called an object of anholonomicity.

We will rewrite the relationship (5.10) in the following manner:

$$\begin{aligned} e^a_{i',j'} &= \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} e^a{}_i + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j} = \\ &= \left(\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Delta^k_{ij}\right) e^a{}_k\,, \end{aligned} \tag{5.15}$$

where we have introduced the notation

$$\Delta^k_{ij} = e^k{}_a e^a{}_{i,j} \tag{5.16}$$

and used the orthogonality condition (5.1).

It is seen from the relationships (5.15) that the object Δ^k_{ij} gets transformed relative to the transformations (5.2) as the connection

$$\Delta^{k'}_{i'j'} = \frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\Delta^k_{ij}. \tag{5.17}$$

The connection of a space given by (5.16) is called the connection of absolute parallelism [31].

Interchanging in (5.17) the indices i and j gives

$$\Delta^{k'}_{j'i'} = \frac{\partial^2 x^k}{\partial x^{j'}\partial x^{i'}}\frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{j'}}\frac{\partial x^j}{\partial x^{i'}}\frac{\partial x^{k'}}{\partial x^k}\Delta^k_{ji}. \tag{5.18}$$

Subtracting (5.18) from (5.17) gives

$$\Delta^{k'}_{[i'j']} = \frac{\partial x^i}{\partial x^{j'}}\frac{\partial x^j}{\partial x^{i'}}\frac{\partial x^{k'}}{\partial x^k}\Delta^k_{[ij]}. \tag{5.19}$$

It follows from the relationships (5.16) and (5.13) that the connection of absolute parallelism features the torsion

$$\Delta^k_{[ij]} = -\Omega_{ij}^{\cdot\cdot k}, \tag{5.20}$$

defined by the object of anholonomity.

5.2 Covariant differentiation in A_4 geometry. Ricci rotation coefficients

The definition of the covariant derivative with respect to the connection of the geometry of absolute parallelism (A_4 geometry) $\Delta^i{}_{jk}$ from a tensor of arbitrary valence $U^{i...p}_{m...n}$ has the form

$$\begin{aligned} \overset{*}{\nabla}_k U^{i...p}_{m...n} = U^{i...p}_{m...n,k} + \Delta^i{}_{jk} U^{j...p}_{m...n} + \ldots + \Delta^p{}_{jk} U^{i...j}_{m...n} - \\ \Delta^j{}_{mk} U^{i...p}_{j...n} - \ldots - \Delta^j{}_{nk} U^{i...p}_{m...j}. \end{aligned} \tag{5.21}$$

This definition enables some quite useful relationships in A_4 geometry to be proved.

Proposition 5.1. Parallel displacement of the tetrad $e^a{}_i$ relative to the connection Δ^i_{jk} equals zero identically.
Proof. From the definition (5.21) we have the following equalities:

$$\overset{*}{\nabla}_k \; e^i{}_a = a^i{}_{a,k} + \Delta^i_{jk} e^j{}_a, \tag{5.22}$$

$$\overset{*}{\nabla}_k \; e^a{}_j = e^a{}_{j,k} - \Delta^i_{jk} e^a{}_i. \tag{5.23}$$

Since the connection Δ^i_{jk} is defined as

$$\Delta^i_{jk} = e^i{}_a e^a{}_{j,k}, \tag{5.24}$$

we have

$$e^i{}_a e^a{}_{j,k} - \Delta^i_{jk} = 0.$$

Multiplying this equality by $e^a{}_i$ and taking into consideration the orthogonality conditions (5.1), we get

$$\overset{*}{\nabla}_k \; e^a{}_j = e^a{}_{j,k} - \Delta^i_{jk} e^a{}_i = 0. \tag{5.25}$$

To prove that the relationship (5.22) is zero, we will take a derivative of the convolution $e^a{}_j e^i{}_a = \delta^i_j$

$$(\delta^i_j)_{,k} = (e^a_j e^i_a)_{,k} = e^i{}_a e^a{}_{j,k} + e^a{}_j e^i_{a,k} = 0.$$

Hence, by (5.24), we have

$$\Delta^i_{jk} = -e^a{}_j e^i{}_{a,k} \tag{5.26}$$

or

$$e^a{}_j e^i{}_{a,k} + \Delta^i_{jk} = 0.$$

Multiplying this relationship by $e^j{}_a$ and using the conditions $e^a{}_j e^i{}_a = \delta^i_j$, we have

$$\overset{*}{\nabla}_k \; e^i{}_a = e^i_{a,k} + \Delta^i_{jk} e^j{}_a = 0. \tag{5.27}$$

Proposition 5.2. Connection Δ^i_{jk} can be represented as the sum

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}, \tag{5.28}$$

where Γ^i_{jk} are the Christoffel symbols given by the relationship (5.6), and

$$T^i_{jk} = -\Omega^{..i}_{jk} + g^{im}(g_{js}\Omega^{..s}_{mk} + g_{ks}\Omega^{..s}_{mj}) \tag{5.29}$$

are the Ricci rotation coefficients [30].
Proof. Let us represent the connection (5.28) as the sum of parts symmetrical and skew-symmetrical in indices j, k

$$\Delta^i_{jk} = \Delta^i_{(jk)} + \Delta^i_{[jk]}, \tag{5.30}$$

where

$$\Delta^i_{(jk)} = \frac{1}{2}(\Delta^i_{jk} + \Delta^i_{kj}), \quad \Delta^i_{[jk]} = \frac{1}{2}(\Delta^i_{jk} - \Delta^i_{jk}).$$

We now add to and subtract from the right-hand side of (5.30) the same expression

$$\Delta^i_{jk} = \Delta^i_{(jk)} + \Delta^i_{[jk]} + g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \tag{5.31}$$

We then group the terms on the right-hand side of (5.31) as follows:

$$\begin{aligned} \Delta^i_{jk} &= \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) + \\ &+ \Delta^i_{[jk]} + g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \end{aligned} \tag{5.32}$$

Since

$$\Delta^i_{[jk]} = -\Omega_{jk}^{\cdot\cdot i},$$

it follows from (5.32) and (5.29) that

$$\Delta^i_{jk} = \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) + T^i_{jk}. \tag{5.33}$$

We now show that

$$\Gamma^i_{jk} = \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \tag{5.34}$$

Actually, we have the relationships

$$\begin{aligned} \Delta^i_{(jk)} &= e^i{}_a e^a_{(j,k)} = \frac{1}{2}e^i{}_a(e^a{}_{j,k} + e^a{}_{k,j}), \\ \Delta^i_{[jk]} &= e^i{}_a e^a_{[j,k]} = \frac{1}{2}e^i{}_a(e^a{}_{j,k} - e^a{}_{k,j}), \\ g_{js} &= \eta_{ab}e^a{}_j e^b{}_s, \end{aligned} \tag{5.35}$$

therefore (5.34) become

$$\begin{aligned} \Gamma^i_{jk} = e^i{}_a e^a_{(j,k)} + g^{im}(\eta_{ab}e^a{}_j e^b_{[m,k]} + \eta_{ab}e^a_k e^b_{[m,j]}) = \\ = \frac{1}{2}\eta^{cd}\eta_{ab}e^i_c e^m_d(e^b_m e^a{}_{j,k} + e^b_m e^c{}_{k,j}) + \\ + \frac{1}{2}g^{im}\left(\eta_{ab}(e^a{}_j e^b{}_{m,k} - e^a{}_j e^b{}_{k,m}) + \eta_{ab}(e^a_k e^b{}_{m,j} - e^a_k e^b{}_{j,m})\right). \end{aligned}$$

Regrouping the terms here gives

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left((\eta_{ab}e^a{}_j e^b{}_m)_{,k} + (\eta_{ab}e^a_k e^b_m)_{,j} - (\eta_{ab}e^a{}_j e^b{}_k)_{,m}\right).$$

Hence, by (5.35), we obtain

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{5.36}$$

or

$$\frac{1}{2}g^{im}(g_{jm,k}+g_{km,j}-g_{jk,m})=$$
$$=\Delta^i_{(jk)}-g^{im}(g_{js}\Delta^s_{[km]}+g_{ks}\Delta^s_{[jm]})=\Gamma^i_{jk}. \qquad (5.37)$$

Substituting (5.37) into (5.33), we get the relationship (5.28).

Proposition 5.3. The Ricci rotation coefficients T^i_{jk} can be represented in the form

$$T^i_{jk}=e^i{}_a\nabla_k e^a{}_j, \qquad (5.38)$$

$$T^i_{jk}=-e^a{}_j\nabla_k e^i{}_a, \qquad (5.39)$$

where ∇_k stands for a covariant derivative with respect to the Christoffel Γ^i_{jk} symbols.

Proof. We will represent in the relationships (5.25) and (5.27) the connection Δ^i_{jk} as the sum (5.28)

$$\overset{*}{\nabla}_k\, e^a{}_j=e^a{}_{j,k}-\Gamma^i_{jk}e^a{}_i-T^i_{jk}e^a{}_i=0, \qquad (5.40)$$

$$\overset{*}{\nabla}_k\, e^i{}_a=e^i_{a,k}+\Gamma^i_{jk}e^j{}_a+T^i_{jk}e^j{}_a=0 \qquad (5.41)$$

Since, by definition [29], we can write

$$\nabla_k e^a{}_j=e^a{}_{j,k}-\Gamma^i_{jk}e^a{}_i,$$

$$\nabla_k e^i{}_a=e^i_{a,k}+\Gamma^i_{jk}e^j{}_a,$$

then (5.40) and (5.41) can be written as

$$\nabla_k e^a{}_j-T^i_{jk}e^a{}_i=0, \qquad (5.42)$$

$$\nabla_k e^i{}_a+T^i_{jk}e^j{}_a=0. \qquad (5.43)$$

Multiplying (5.42) by $e^i{}_a$ and (5.43) by $e^a{}_j$, respectively, we will obtain (using the orthogonality conditions (5.1)), by (5.42), (5.43), the relationships (5.38) and (5.39).

We will now calculate the covariant derivative $\overset{*}{\nabla}_k$ with respect to the metric tensor g^{jm}, knowing that $g^{jm}=\eta^{ab}e^j{}_a e^m_b$

$$\overset{*}{\nabla}_k\, g^{jm}=\overset{*}{\nabla}_k\,\eta^{ab}e^j{}_a e^m_b=\overset{*}{\nabla}_k\, e^j{}_a e^{ma}=$$
$$=e^{ma}\,\overset{*}{\nabla}_k\, e^j{}_a+e^j{}_a\,\overset{*}{\nabla}_k\, e^{ma}.$$

From the relationships (5.25) and (5.27), we have

$$\overset{*}{\nabla}\, g^{jm}=0. \qquad (5.44)$$

On the other hand, applying the formula (5.21) to the relationship (5.44), we find that

$$\overset{*}{\nabla}_k\, g^{jm} = g^{jm}_{,k} + \Delta^j_{pk} g^{pm} + \Delta^m_{pk} g^{jp} = 0. \qquad (5.45)$$

Substituting the connection Δ^i_{jk} as the sum (5.28), we will write the relationship (5.45) in the form

$$\overset{*}{\nabla}_k\, g^{jm} = \nabla_k g^{jm} + T^j_{pk} g^{pm} + T^m_{pk} g^{jp} = 0. \qquad (5.46)$$

From the equality

$$\nabla_k g^{jm} = g^{jm}_{,k} + \Gamma^j_{pk} g^{pm} + \Gamma^m_{pk} g^{jp} = 0, \qquad (5.47)$$

we have, by (5.46),

$$T^j_{pk} g^{pm} + T^m_{pk} g^{jp} = T^{jm}_{k} + T^{mj}_{k} = 0.$$

This equality establishes the following symmetry properties for the Ricci rotation coefficients:

$$T_{jmk} = -T_{mjk}. \qquad (5.48)$$

Therefore, in the A_4 geometry the Ricci rotation coefficients have 24 independent components.

5.3 Curvature tensor of A_4 space

The curvature tensor of the space of absolute parallelism $S^i{}_{jkm}$ is defined in terms of the connection Δ^i_{jk} following a conventional rule [18]

$$S^i{}_{jkm} = 2\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]} = 0, \qquad (5.49)$$

where the parentheses [ ] signify alternation in appropriate indices, whereas the index within the vertical lines | | is not subject to alternation.

Proposition 5.4. The Riemann-Christoffel tensor of a space with the connection (5.26) equals zero identically.
Proof. From the relationship (5.26) we have

$$e^a{}_{j,k} = \Delta^i_{jk} e^a{}_i. \qquad (5.50)$$

Differentiating the relationship (5.50) with respect to m gives

$$e^a{}_{j,k,m} = (\Delta^i_{jk} e^a{}_i)_{,m} = \Delta^i_{jk,m} e^a{}_i + e^a_{i,m}\Delta^i_{jk} =$$
$$= (\Delta^i_{jk,m} + e^i{}_a e^a_{s,m}\Delta^s_{jk}) e^a{}_i = (\Delta^i_{jk,m} + \Delta^i_{sm}\Delta^s_{jk}) e^a{}_i.$$

Alternating this relationship in indices k and m we get

$$-2e^a_{j,[k,m]} = 2(\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]}) = S^i{}_{jkm} e^a{}_i. \qquad (5.51)$$

Since the operation of differentiating with respect to indices k and m is symmetrical, we have

$$e^a_{j,[k,m]} = 0,$$

From this equality, considering that $e^a{}_i$ in (5.51) is arbitrary, we will get

$$S^i{}_{jkm} = 0. \tag{5.52}$$

Proposition 5.5. Tensor $S^i{}_{jkm}$ can be represented as the sum

$$S^i{}_{jkm} = R^i{}_{jkm} + 2\nabla_{[k}T^i_{|j|m]} + 2T^i_{c[k}T^c_{|j|m]} = 0, \tag{5.53}$$

where

$$R^i{}_{jkm} = 2\Gamma^i_{j[m,k]} + 2\Gamma^i_{s[k}\Gamma^s_{|j|m]} \tag{5.54}$$

is the tensor of the Riemannian space A_4.

Proof. Substituting the sum $\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}$ into (5.49) gives

$$S^i{}_{jkm} = 2\Gamma^i_{j[m,k]} + 2\Gamma^i_{s[k}\Gamma^s_{|j|m]} + 2T^i_{j[m,k]} + 2T^i_{s[k}T^s_{|j|m]} + \\ 2T^i_{s[k}\Gamma^s_{|j|m]} + 2\Gamma^i_{s[k}T^s_{|j|m]} = 0. \tag{5.55}$$

Using (5.54), we will write (5.55) as follows:

$$S^i{}_{jkm} = R^i{}_{jkm} + 2T^i_{j[m,k]} + 2T^i_{s[k}T^s_{|j|m]} + \\ +2\Gamma^s_{j[k}T^i_{|s|m]} + 2\Gamma^i_{s[k}T^s_{|j|m]} = 0. \tag{5.56}$$

If now we add to the right-hand side of this relationship the expression

$$-2\Gamma^s_{[km]}T^i_{sj} = 0,$$

and take into consideration that [29]

$$\nabla_k U^{i...p}_{m...n} = U^{i...p}_{m...n,k} + \Gamma^i_{jk}U^{j...p}_{m...n} + \ldots + \Gamma^p_{jk}U^{i...j}_{m...n} - \\ \Gamma^j_{mk}U^{i...p}_{j...n} - \ldots - \Gamma^j_{nk}U^{i...p}_{m...j}, \tag{5.57}$$

we will obtain from (5.56) the equality (5.53).

Let us now rewrite the relationship (5.53) as

$$R^i{}_{jkm} = -2T^i_{j[m,k]} - 2T^i_{s[k}T^s_{|j|m]}. \tag{5.58}$$

Substituting here (5.38) and (5.39)

$$T^i_{jk} = e^i{}_a\nabla_k e^a{}_j, \quad T^i_{jk} = -e^a{}_j\nabla_k e^i{}_a,$$

we obtain

$$-2T^i_{j[m,k]} = -2e^i{}_a\nabla_{[k}\nabla_{m]}e^a{}_j - 2\nabla_{[k}e^i_{|a|}\nabla_{m]}e^a{}_j,$$

$$-2T^i_{s[k}T^s_{|j|m]} = 2e^a{}_s\nabla_{[k}e^i_{|a}e^s{}_{a|}\nabla_{m]}e^a{}_j = 2\nabla_{[k}e^i_{|a|}\nabla_{m]}e^a{}_j.$$

Therefore, it follows from the relationships (5.58) that

$$R^{i}{}_{jkm} = -2e^{i}{}_{a}\nabla_{[k}\nabla_{m]}e^{a}{}_{j} = 2e^{i}{}_{a}\nabla_{[m}\nabla_{k]}e^{a}{}_{j}. \tag{5.59}$$

Proposition 5.6. The torsion field $\Omega_{jk}^{\cdot\cdot i}$ of the A_4 space satisfies the equations

$$\overset{*}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} + 2\Omega_{[kj}^{\cdot\cdot s}\Omega_{m]s}^{\cdot\cdot i} = 0. \tag{5.60}$$

Proof. Alternating the expression (5.49) in indices j,k,m and using the relationship $\Delta^{i}_{[jk]} = -\Omega_{jk}^{\cdot\cdot i}$, we get

$$S^{i}_{[jkm]} = 2\Omega_{[jm,k]}^{\cdot\cdot i} + 2\Delta^{i}{}_{s[k}\Omega_{jm]}^{\cdot\cdot s} = 0. \tag{5.61}$$

If then we add and subtract here the quantity

$$2\Delta^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} + 2\Delta^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i},$$

we will have

$$2\Omega_{[jm,k]}^{\cdot\cdot i} + 2\Delta^{i}{}_{s[k}\Omega_{jm]}^{\cdot\cdot s} - 2\Delta^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} - 2\Delta^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i} + \\ 2\Delta^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} + 2\Delta^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i} = 0.$$

Using the formula (5.21), we can rewrite this relationship as follows:

$$\begin{aligned} 2\,\overset{*}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} - 2\Omega_{[kj}^{\cdot\cdot s}\Omega_{|s|m]}^{\cdot\cdot i} - 2\Omega_{[km}^{\cdot\cdot s}\Omega_{j]s}^{\cdot\cdot i} = \\ = 2\,\overset{*}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} + 4\Omega_{[kj}^{\cdot\cdot s}\Omega_{m]s}^{\cdot\cdot i} = 0, \end{aligned} \tag{5.62}$$

whence we have (5.60).

Proposition 5.7. The Riemann tensor $R^{i}{}_{jkm}$ of the A_4 space satisfies the equality

$$R^{i}_{[jkm]} = 0. \tag{5.63}$$

Proof. Alternating the relationship (5.54) in indices j,k,m and using the equality

$$T^{i}_{[jk]} = -\Omega_{jk}^{\cdot\cdot i},$$

we have

$$R^{i}_{[jkm]} = 2\nabla_{[k}\Omega_{jm]}^{\cdot\cdot i} + 2T^{i}{}_{s[k}\Omega_{jm]}^{\cdot\cdot s}.$$

If in the right-hand side of the equality we add and subtract the quantity

$$2T^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} + 2T^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i},$$

we obtain

$$\begin{aligned} R^{i}_{[jkm]} = 2\nabla_{[k}\Omega_{jm]}^{\cdot\cdot i} + 2T^{i}{}_{s[k}\Omega_{jm]}^{\cdot\cdot s} - 2T^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} - 2T^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i} + \\ +2T^{s}_{[kj}\Omega_{|s|m]}^{\cdot\cdot i} + 2T^{s}_{[km}\Omega_{j]s}^{\cdot\cdot i} = 2\,\overset{*}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} - 2\Omega_{[kj}^{\cdot\cdot s}\Omega_{|s|m]}^{\cdot\cdot i} - \\ -2\Omega_{[km}^{\cdot\cdot s}\Omega_{j]s}^{\cdot\cdot i} = 2\,\overset{*}{\nabla}_{[k}\,\Omega_{jm]}^{\cdot\cdot i} + 4\Omega_{[kj}^{\cdot\cdot s}\Omega_{m]s}^{\cdot\cdot i} = 0, \end{aligned}$$

which proves the validity of the relationship (5.63).

5.4 Formalism of external forms and the matrix treatment of Cartan's structural equations of the absolute parallelism geometry

Consider the differentials

$$dx^i = e^a e^i{}_a, \tag{5.64}$$

$$de^i{}_b = \Delta^a{}_b e^i{}_a, \tag{5.65}$$

where

$$e^a = e^a_i dx^i, \tag{5.66}$$

$$\Delta^a{}_b = e^a{}_i de^i{}_b = \Delta^a{}_{bk} dx^k \tag{5.67}$$

are differential 1-forms of tetrad $e^a{}_i$ and connection of absolute parallelism $\Delta^a{}_{bk}$. Differentiating the relationships (5.64), (5.65) externally [31], we have, respectively,

$$d(dx^i) = (de^a - e^c \wedge \Delta^a{}_c)e^i{}_a = -S^a e^i{}_a, \tag{5.68}$$

$$d(de^i{}_a) = (d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c)e^i{}_b = -S^b{}_a e^i{}_b. \tag{5.69}$$

Here S^a denotes the 2-form of Cartanian torsion [31], and $S^b{}_a$ – the 2-form of the curvature tensor. The sign $\wedge$ signifies external product, e.g,

$$e^a \wedge e^b = e^a e^b - e^b e^a. \tag{5.70}$$

By definition, a space has a geometry of absolute parallelism, if the 2-form of Cartanian torsion S^a and the 2-form of the Riemann-Christoffel curvature $S^b{}_a$ of this space vanishe

$$S^a = 0, \tag{5.71}$$

$$S^b{}_a = 0. \tag{5.72}$$

At the same time, these equalities are the integration conditions for the differentials (5.64) and (5.65).

Equations

$$de^a - e^c \wedge \Delta^a{}_c = -S^a, \tag{5.73}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = -S^b{}_a, \tag{5.74}$$

which follow from (5.68) and (5.69), are Cartan's structural equations for an appropriate geometry. For the geometry of absolute parallelism hold the conditions (5.71) and (5.72), therefore Cartan's structural equations for A_4 geometry have the form

$$de^a - e^c \wedge \Delta^a{}_c = 0, \tag{5.75}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = 0. \tag{5.76}$$

Considering (5.28), we will represent 1-form $\Delta^a{}_b$ as the sum

$$\Delta^a{}_b = \Gamma^a{}_b + T^a{}_b. \tag{5.77}$$

Substituting this relationship into (5.75) and noting that

$$e^c \wedge \Delta^a{}_c = e^c \wedge T^a{}_c,$$

we get the first of Cartan's structural equations for A_4 space.

$$de^a - e^c \wedge T^a{}_c = 0. \tag{A}$$

Substituting (5.77) into (5.76) gives the second of Cartan's equations for A_4 space.

$$R^a_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c = 0, \tag{B}$$

where $R^a{}_b$ is the 2-form of the Riemann tensor

$$R^a{}_b = d\Gamma^a{}_b - \Gamma^c{}_b \wedge \Gamma^a{}_c. \tag{5.78}$$

By definition [31], we always have the relationships

$$dd(dx^i) = 0, \tag{5.79}$$

$$dd(de^i{}_a) = 0. \tag{5.80}$$

In the geometry of absolute parallelism these equalities become

$$d(de^a - e^c \wedge T^a{}_c) = R^a{}_{cfd} e^c \wedge e^f \wedge e^d = 0, \tag{5.81}$$

$$d(R^a{}_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c) = dR^a{}_b + R^f{}_b \wedge T^a{}_f - T^f{}_b \wedge R^a{}_f = 0. \tag{5.82}$$

Here

$$R^a{}_{cfd} = -2T^a{}_{c[d,f]} - 2T^a{}_{b[f}T^b{}_{|c|d]}.$$

Equalities (5.81) and (5.82) represent the first and second of Bianchi's identities, respectively, for A_4 space. Dropping the indices, we can write Cartan's structural equations and Bianchi's identities for the A_4 geometry as

$$\boxed{\begin{array}{cr} de - e\wedge T = 0, & (A)\\ R + dT - T\wedge T = 0, & (B)\\ R\wedge e\wedge e\wedge e = 0, & (C)\\ dR + R\wedge T - T\wedge R = 0. & (D)\end{array}}$$

Proposition 5.8. The matrix treatment of the first of Cartan's structural equations (A) of the A_4 geometry has the form

$$\nabla_{[k} e^a{}_{m]} - e^b{}_{[k} T^a{}_{|b|m]} = 0. \tag{5.83}$$

Proof. Let us write equations (A) as

$$de^a - e^c \wedge T^a_{\ c} = 0. \tag{5.84}$$

Further, by (5.66), we have

$$de^a = d(e^a_{\ m} dx^m) = \nabla_k e^a_{\ m} dx^k \wedge dx^m = \frac{1}{2}(\nabla_k e^a_{\ m} - \nabla_m e^a_{\ k}) dx^k \wedge dx^m$$

and, also,

$$e^b \wedge T^a_{\ b} = e^b_{\ k} T^a_{\ bm} dx^k \wedge dx^m = \frac{1}{2}(e^b_{\ k} T^a_{\ bm} - e^b_{\ m} T^a_{\ bk}) dx^k \wedge dx^m.$$

Substituting these relationships into equations (5.84) we will derive the matrix equations in the form

$$\nabla_{[k} e^a_{\ m]} - e^b_{\ [k} T^a_{\ |b|m]} = 0, \tag{A}$$

where the matrixes $e^a_{\ m}$ and $T^a_{\ bm}$ in world indices $i, j, m, \ldots$ are transformed as vectors

$$e^a_{\ m'} = \frac{\partial x^m}{\partial x^{m'}} e^a_{\ m}, \tag{5.85}$$

$$T^a_{\ bm'} = \frac{\partial x^m}{\partial x^{m'}} T^a_{\ bm}, \tag{5.86}$$

and in the matrix indices $a, b, c, \ldots$ they are transformed as follows:

$$e^{a'}_{\ m} = \Lambda_a^{\ a'} e^a_{\ m}, \tag{5.87}$$

$$T^{a'}_{\ b'k} = \Lambda_a^{\ a'} T^a_{\ bk} \Lambda^b_{\ b'} + \Lambda^{a'}_{\ a} \Lambda^a_{\ b',k}. \tag{5.88}$$

In relationships (5.87) and (5.88) the matrices $\partial x^{m'}/\partial x^m$ form a translation group T_4 that is defined on a manifold of world coordinates x^i. On the other hand, the matrices $\Lambda^{a'}_a$ form a group of four-dimensional rotations $O(3.1)$

$$\Lambda^{a'}_a \in O(3.1),$$

defined on the manifold of "angular coordinates" $e^a_{\ \ i}$. Actually, the tetrad $e^a_{\ \ i}$ is a mathematical image of an arbitrarily accelerated four-dimensional reference frame. Such a frame has ten degrees of freedom: four translational ones connected with the motion of its origin, and six angular ones describing variations of its orientation. The six independent components of the tetrad $e^a_{\ \ i}$ represent six direction cosines of six independent angles defining the orientation of the tetrad in space.

Proposition 5.9. The matrix rendering of the second of Cartan's structuring equations (B) of the A_4 geometry has the form

$$R^a_{\ bkm} + 2\nabla_{[k} T^a_{|b|m]} + 2T^a_{\ c[k} T^c_{\ |b|m]} = 0. \tag{5.89}$$

Proof. We will expand the 2-form $R^a{}_d$ as

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \frac{1}{2} R^a{}_{bkm} dx^k \wedge dx^m . \tag{5.90}$$

Further, we have

$$dT^a_{\ b} = d(T^a_{\ bm} dx^m) = \nabla_k T^a_{\ bm} dx^k \wedge dx^m = \frac{1}{2}(\nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk}) dx^k \wedge dx^m , \tag{5.91}$$

and also

$$T^a_{\ c} \wedge T^c_{\ b} = T^a_{\ ck} T^c_{\ bm} dx^k \wedge dx^m = = \frac{1}{2}(T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck}) dx^k \wedge dx^m . \tag{5.92}$$

Let us substitute the relationships (5.92)–(5.94) into

$$R^a{}_b + dT^a_{\ b} - T^c_{\ b} \wedge T^a_{\ c} = 0.$$

Simple transformations yield

$$\frac{1}{2}(R^a{}_{bkm} + \nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk} + T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck}) dx^k \wedge dx^m = 0.$$

Since here the factor $dx^k \wedge dx^m$ is arbitrary, we have

$$R^a{}_{bkm} + \nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk} + T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck} = 0,$$

which is equivalent to the equations (5.89).

Proposition 5.10. The matrix form of the Bianchi identity (D) of A_4 geometry is

$$\nabla_{[n} R^a{}_{|b|km]} + R^c{}_{b[km} T^a_{\ |c|n]} - T^c_{\ b[n} R^a{}_{|c|km]} = 0. \tag{5.93}$$

Proof. The external differential $dR^a{}_b$ in the identities (D) has the 2-form

$$dR^a{}_b = \frac{1}{2} \nabla_n R^a{}_{bkm} dx^n \wedge dx^k \wedge dx^m = = \frac{1}{6}(\nabla_n R^a{}_{bkm} + \nabla_m R^a{}_{bkn} + \nabla_k R^a{}_{bmn}) dx^n \wedge dx^k \wedge dx^m . \tag{5.94}$$

In addition, we have

$$R^f{}_b \wedge T^a_{\ f} = \frac{1}{2} R^f{}_{bkm} T^a_{\ fn} dx^k \wedge dx^m \wedge dx^n = = \frac{1}{6}(R^f{}_{bkm} T^a_{\ fn} + R^f{}_{bnk} T^a_{\ fm} + R^f{}_{bmn} T^a_{\ fk}) dx^k \wedge dx^m \wedge dx^n , \tag{5.95}$$

$$T^f_{\ b} \wedge R^a{}_f = \frac{1}{2} T^f_{\ bn} R^a{}_{fkm} dx^n \wedge dx^k \wedge dx^m = = \frac{1}{6}(T^f_{\ bn} R^a{}_{fkm} + T^f_{\ bm} R^a{}_{fnk} + T^f_{\ bk} R^a{}_{fmn}) dx^n \wedge dx^k \wedge dx^m . \tag{5.96}$$

Substituting relationships (5.94)–(5.96) into the identity

$$dR^a{}_b + R^f{}_b \wedge T^a{}_f - T^f{}_b \wedge R^a{}_f = 0$$

and considering that $dx^n \wedge d^k \wedge dx^m$ is arbitrary, we get

$$\nabla_n R^a{}_{bkm} + \nabla_m R^a{}_{bkn} + \nabla_k R^a{}_{bmn} + R^f{}_{bkm} T^a{}_{fn} + R^f{}_{bnk} T^a{}_{fm} +$$
$$+ R^f{}_{bmn} T^a{}_{fk} - T^f{}_{bn} R^a{}_{fkm} - T^f{}_{bm} R^a{}_{fnk} - T^f{}_{bk} R^a{}_{fmn} = 0,$$

which is equivalent to the identity (5.93).

The first of Bianchi's identities (C) of A_4 geometry in indices of the group $O(3.1)$ is written as

$$R^a{}_{[bcd]} = 0, \tag{5.97}$$

or, which is the same, as

$$\overset{*}{\nabla}_{[b}\, \Omega_{cd]}^{\cdot\cdot a} + 2\Omega_{[bc}^{\cdot\cdot f}\Omega_{d]f}^{\cdot\cdot a} = 0. \tag{5.98}$$

5.5 A$_4$ geometry as a group manifold. Killing-Cartan metric

The matrix representation of Cartan's structural equations of the geometry of absolute parallelism indicates that, in fact, this space behaves as a manifold, on which the translations group T_4 and the rotations group $O(3.1)$ are specified. We will consider A_4 geometry as a group 10-dimensional manifold formed by four translational coordinates x_i ($i = 0, 1, 2, 3$) and six (by the relationship $e^a{}_i e^j{}_a = \delta_i{}^j$) angular coordinates $e^a{}_i$ ($a = 0, 1, 2, 3$). Suppose that on this manifold a group of four-dimensional translations T_4 and a rotations group $O(3.1)$ are defined. We then introduce the Hayashi invariant derivative [32]

$$\nabla_b = e^k{}_b \partial_k, \tag{5.99}$$

whose components are generators of the translations group T_4 that is specified on the manifold of translational coordinates x_i. If then we represent as a sum

$$e^k{}_b = \delta^k{}_b + a^k{}_b, \tag{5.100}$$

$$i, j, k \ldots = 0, 1, 2, 3, \quad a, b, c, \ldots = 0, 1, 2, 3,$$

then the field $a^k{}_b$ can be viewed as the potential of the gauge field of the translations group T_4 [32]. In the case where $a^k{}_b = 0$, the generators (5.99) coincide with the generators of the translations group of the pseudo-Euclidean space E_4 .

We know already that in the coordinate index k the nonholonomic tetrad $e^k{}_a$ transforms as the vector

$$e^{k'}{}_a = \frac{\partial x^{k'}}{\partial x^k} e^k{}_a,$$

whence, by (5.100), we have the law of transformation for the field $a^k{}_a$ relative to the translationss

$$a^{k'}{}_b = \frac{\partial x^{k'}}{\partial x^n} a^n{}_b + \frac{\partial x^{k'}}{\partial x^n} \delta^n{}_b - \delta^{k'}_b. \qquad (5.101)$$

We define the tetrad $e^i{}_a$ as

$$e^i{}_a = \nabla_a x^i \qquad (5.102)$$

and write the commutational relationships for the generators (5.99) as

$$\nabla_{[a} \nabla_{b]} = -\Omega_{ab}^{..c} \nabla_c , \qquad (5.103)$$

where $-\Omega_{ab}^{..c}$ are the structural functions for the translations group of the space A_4. If then we apply the operator (5.103) to the manifold x^i, we will arrive at the structural equations of the group T_4 of the space A_4 as

$$\nabla_{[a} \nabla_{b]} x^i = -\Omega_{ab}^{..c} \nabla_c x^i \qquad (5.104)$$

or

$$\nabla_{[a} e^i{}_{b]} = -\Omega_{ab}^{..c} e^i{}_c . \qquad (5.105)$$

In this relationship the structural functions $-\Omega_{ab}^{..c}$ are defined as

$$-\Omega_{ab}^{..c} = e^c{}_i \nabla_{[a} e^i{}_{b]}. \qquad (5.106)$$

It is seen from this equality that when the potentials of the gauge field of translations group a^k_b in the relationship (5.100) vanish, so do the structural functions (5.106). Therefore, we will refer to the field $\Omega_{ab}^{..c}$ as the gauge field of the translations group.

Considering that $T^c{}_{[ab]} = -\Omega_{ab}^{..c}$, we will rewrite the structural equations (5.106) as

$$\nabla_{[k} e^a{}_{m]} - e^b{}_{[k} T^a{}_{|b|m]} = 0. \qquad (5.107)$$

It is easily seen that the equations (5.107) can be derived by alternating the equations (5.42). What is more, they coincide with the structural Cartan equations (A) of the geometry of absolute parallelism.

The structural equations of group T_4, written as (5.106), can be regarded as a definition for the torsion of space A_4. So the torsion of space A_4 coincides with the structural function of the translations group of this space, such that the structural functions obey the generalized Jacobi identity

$$\overset{*}{\nabla}_{[b}\, \Omega_{cd]}^{..a} + 2\Omega_{[bc}^{..f} \Omega_{d]f}^{..a} = 0, \qquad (5.108)$$

where $\overset{*}{\nabla}_b$ is the covariant derivative with respect to the connection of absolute parallelism Δ^a_{bc}. Comparing the identity (5.108) with the Bianchi identity (5.98) of the geometry A_4, we see that we deal with the same identity. The Jacobi identity (5.108), which is obeyed by the structural functions of the translations group of geometry A_4, coincides with the first Bianchi identity of the geometry of absolute parallelism .

The vectors

$$e^i{}_a = \nabla_a x^i, \tag{5.109}$$

that form the vector stratification [31] of the A_4 geometry, point along the tangents to each point of the manifold x^i of the pseudo-Euclidean plane with the metric tensor

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1, -1, -1, -1). \tag{5.110}$$

Therefore, the ten-dimensional manifold (four translational coordinates x^i and six "rotational" coordinates $e^i{}_a$) of the geometry of absolute parallelism can be regarded as the stratification with the coordinates of the base x^i and the (anholonomic) "coordinates" of the fibre $e^i{}_c$. If on the base x^i we have the translations group T_4, then in the fibre $e^i{}_c$ we have the rotation group $O(3.1)$. It follows from (5.109) that the infinitesimal translations in the base x^i in the direction a are given by the vector

$$ds^a = e^a{}_i dx^i. \tag{5.111}$$

If from (5.111) and the covariant vector $ds_a = e^i{}_a dx_i$ we form the invariant convolution ds^2, we will obtain the Riemannian metric of A_4 space

$$ds^2 = g_{ik} dx^i dx^k \tag{5.112}$$

with the metric tensor

$$g_{ik} = \eta_{ab} e^a{}_i e^b{}_k.$$

Therefore, the Riemannian metric (5.112) can be viewed as the metric defined on the translations group T_4.

Since in the fibre we have the "angular coordinates" $e^i{}_a$ that form a manifold in which group $O(3.1)$ is defined, then it would be natural to define the structural equations for this group, as well as the metric specified on the group $O(3.1)$.

Let us rewrite the relationships (5.38) and (5.39) in matrix form

$$T^a{}_{bk} = e^a{}_i T^i{}_{jk} e^j{}_b = \nabla_k e^a{}_j e^j{}_b, \tag{5.113}$$

$$T^a{}_{bk} = e^a{}_i T^i{}_{jk} e^j{}_b = -e^a{}_i \nabla_k e^i{}_b. \tag{5.114}$$

These relationships enable the dependence between the infinitesimal rotation $d\chi_{ab} = -d\chi_{ba}$ of the vector $e^a{}_i$ at infinitesimal translations ds_a to be established. In fact, by (5.113) and (5.114), we have

$$d\chi^a{}_b = T^a{}_{bk} dx^k = De^a{}_j e^j{}_b, \tag{5.115}$$

$$d\chi^a{}_b = T^a{}_{bk} dx^k = -e^a{}_i De^i{}_b. \tag{5.116}$$

where D is the absolute differential [29] with respect to the Christoffel symbols Γ^i_{jk}. Using (5.115), we can form the invariant quadratic form $d\tau^2 = d\chi^a{}_b d\chi^b{}_a$ to arrive at the Killing-Cartan metric

$$d\tau^2 = d\chi^a{}_b d\chi^b{}_a = T^a{}_{bk} T^b{}_{an} dx^k dx^n = -De^a{}_i De^i{}_a \tag{5.117}$$

with the metric tensor

$$H_{kn} = T^a{}_{bk} T^b{}_{an}. \tag{5.118}$$

Unlike metric (5.112), the metric (5.117) is specified on the rotations group $O(3.1)$ that acts on the manifold of the "rotational coordinates" $e^a{}_i$.

Let us now introduce the covariant derivative

$$\overset{*}{\nabla}_m = \nabla_m + T_m, \tag{5.119}$$

where T_m is the matrix $T^a{}_{bm}$ with discarded matrix indices. We will regard the components of the derivative as generators of the rotations group $O(3.1)$. Applying this operator to the tetrad e^i that forms the manifold of "angular coordinates" of the A_4 geometry, we will arrive at

$$\overset{*}{\nabla}_m \, e^i = \nabla_m e^i + T_m e^i = 0, \tag{5.120}$$

hence

$$T_m = -e_i \nabla_m e^i. \tag{5.121}$$

It is interesting to note that, just as in (5.109) we have defined six "angular coordinates" $e^i{}_a$ through the four translational coordinates x^i, so in (5.121) we can define 24 "supercoordinates" $T^a{}_{bm}$ through the six coordinates $e^i{}_a$.

It follows from (5.120) that

$$\nabla_m e^i = -T_m e^i. \tag{5.122}$$

Recall that in the relationships (5.120)-(5.122) we have defined through ∇_m the covariant derivative with respect to Γ^i_{jk}. We will now take the covariant derivative ∇_k of the relationships (5.122)

$$\begin{aligned} \nabla_k \nabla_m e^i = -\nabla_k (T_m e^i) = -(\nabla_k T_m e^i + T_m \nabla_k e^i) = \\ = -(\nabla_k T_m e^i + T_m e^i e_i \nabla_k e^i). \end{aligned}$$

Using (5.121), we will rewrite this expression as follows

$$\nabla_k \nabla_m e^i = -(\nabla_k T_m - T_m T_k) e^i.$$

Alternating this expression in the indices k and m gives

$$\nabla_{[k} \nabla_{m]} e^i = \frac{1}{2} R_{km} e^i, \tag{5.123}$$

where

$$R_{km} = 2\nabla_{[m} T_{k]} + [T_m, T_k]. \tag{5.124}$$

Introducing in equations (5.124) the matrix indices (the fibre indices), we will obtain the structural equation of the group $O(3.1)$

$$R^a{}_{bkm} = 2\nabla_{[m} T^a{}_{|b|k]} + 2T^a{}_{c[m} T^c{}_{|b|k]}. \tag{B}$$

It is easily seen that the structural equations of the rotations group (B) coincide with the second of Cartan's structural equations (5.124) of the geometry A_4.

In this case the quantities $T^a_{\ bk}$ and $R^a_{\ bkm}$ vary in the rotations group $O(3.1)$ following the law

$$T^{a'}_{\ b'k} = \Lambda_a^{\ a'} T^a_{\ bk} \Lambda^b_{\ b'} + \Lambda_a^{\ a'} \Lambda^a_{\ b',k}, \tag{5.125}$$

and appear as the potentials of the gauge field $R^a_{\ bkm}$ of the rotations group $O(3.1)$. In the process, the gauge field of the group $O(3.1)$ obeys the formula

$$R^{a'}_{\ b'km} = \Lambda_a^{\ a'} R^a_{\ bkm} \Lambda^b_{\ b'}. \tag{5.126}$$

Note that the structural functions of the rotations group of A_4 geometry are the components of the curvature tensor $R^a_{\ bkm}$. It can be shown that the structural functions $R^a_{\ bkm}$ of the rotations group $O(3.1)$ satisfy the Jacobi identity

$$\nabla_{[n} R^a_{\ |b|km]} + R^c_{\ b[km} T^a_{\ |c|n]} - T^c_{\ b[n} R^a_{\ |c|km]} = 0, \tag{D}$$

which, at it was shown in the previous section, are at the same time the second Bianchi identities of the A_4 space.

Let us introduce the dual Riemann tensor

$$\overset{*}{R}_{ijkm} = \frac{1}{2} \varepsilon^{sp}_{\ \ km} R_{ijsp}, \tag{5.127}$$

where $\varepsilon^{sp}_{\ \ km}$ is the completely skew-symmetrical Levi-Chivita tensor. Then the equations (D) can be written as

$$\nabla_n \overset{*}{R}{}^a_{\ b}{}^{kn} + \overset{*}{R}{}^c_{\ b}{}^{kn} T^a_{\ cn} - T^c_{\ bn} \overset{*}{R}{}^a_{\ c}{}^{kn} = 0 \tag{5.128}$$

or, if we drop the matrix indices, as

$$\nabla_n \overset{*}{R}{}^{kn} + \overset{*}{R}{}^{kn} T_n - T_n \overset{*}{R}{}^{kn} = 0. \tag{5.129}$$

5.6 Structural equations of A$_4$ geometry in the form of expanded, completely geometrized Einstein-Yang-Mills set of equations

Einstein believed that one of the main problems of the unified field theory was the geometrization of the energy-momentum tensor of matter on the right-hand side of his equations. This problem can be solved if we use as the space of events the geometry of absolute parallelism and the structural Cartan equations for this geometry.

In fact, folding the equations (B), written as

$$R^i_{\ jkm} + 2\nabla_{[k} T^i_{|j|m]} + 2T^i_{s[k} T^s_{|j|m]} = 0 \tag{5.130}$$

in indices i and k, gives

$$R_{jm} = -2\nabla_{[i} T^i_{|j|m]} - 2T^i_{s[i} T^s_{|j|m]}. \tag{5.131}$$

If then we fold the equations (5.131) with the metric tensor g^{jm}, we have

$$R = -2g^{jm}(\nabla_{[i}T^{i}_{|j|m]} + 2T^{i}_{s[i}T^{s}_{|j|m]}). \tag{5.132}$$

Forming, using (5.131) and (5.132), the Einstein tensor

$$G_{jm} = R_{jm} - \frac{1}{2}g_{jm}R,$$

we obtain the equations

$$R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, \tag{5.133}$$

which are similar to Einstein's equations, but with the geometrized right-hand side defined as

$$T_{jm} = -\frac{2}{\nu}\{(\nabla_{[i}T^{i}{}_{|j|m]} + T^{i}{}_{s[i}T^{s}{}_{|j|m]}) - \\ -\frac{1}{2}g_{jm}g^{pn}(\nabla_{[i}T^{i}{}_{|p|n]} + T^{i}{}_{s[i}T^{s}{}_{|p|n]})\} \tag{5.134}$$

Using the notation

$$P_{jm} = (\nabla_{[i}T^{i}{}_{|j|m]} + T^{i}{}_{s[i}T^{s}{}_{|j|m]})$$

then, by (5.134), we have

$$T_{jm} = -\frac{2}{\nu}(P_{jm} - \frac{1}{2}g_{jm}g^{pn}P_{pn}). \tag{5.135}$$

Tensor (5.135) has parts that are both symmetrical and skew-symmetrical in indices j and m, i.e.,

$$T_{jm} = T_{(jm)} + T_{[jm]}. \tag{5.136}$$

The left-hand side of the equations (5.133) is always symmetrical in indices j and m, therefore these equations can be written as

$$R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{(jm)}, \tag{5.137}$$

$$T_{[jm]} = \frac{1}{\nu}(-\nabla_i\Omega^{..i}_{jm} - \nabla_m A_j - A_s\Omega^{..s}_{jm}) = 0, \tag{5.138}$$

where

$$A_j = T^{i}_{ji}. \tag{5.139}$$

Relationship (5.138) can be taken to be the equations obeyed by the torsion fields $\Omega^{..i}_{jm}$, which form the energy-momentum tensor (5.135).

In the case where the field T^{i}_{jk} is skew-symmetrical in all the three indices, we get

$$T_{ijk} = -T_{jik} = T_{jki} = -\Omega_{ijk}. \tag{5.140}$$

For such fields the equations (5.138) become simple, namely

$$\nabla_i \Omega^{\cdot\cdot i}_{jm} = 0. \tag{5.141}$$

The energy-momentum tensor (5.135) is symmetrical in indices j, m and appears to be given by

$$T_{jm} = \frac{1}{\nu}(\Omega^{\cdot\cdot i}_{sm}\Omega^{\cdot\cdot s}_{ji} - \frac{1}{2}g_{jm}\Omega^{\cdot ji}_{s}\Omega^{\cdot\cdot s}_{ji}). \tag{5.142}$$

By (5.137), we have

$$T_{jm} = \frac{1}{\nu}(R_{jm} - \frac{1}{2}g_{jm}R). \tag{5.143}$$

Using (5.131), (5.140) and (5.142) gives

$$R_{jm} = \Omega^{\cdot\cdot i}_{sm}\Omega^{\cdot\cdot s}_{ji}, \tag{5.144}$$

$$R = g^{jm}\Omega^{\cdot\cdot i}_{sm}\Omega^{\cdot\cdot s}_{ji} = \Omega^{\cdot ji}_{s}\Omega^{\cdot\cdot s}_{ji}. \tag{5.145}$$

Substituting (5.144) and (5.145) into (5.143), we arrive at the energy-momentum tensor (5.142).

Through the field (5.140) we can define the pseudo-vector h_m as follows

$$\Omega^{ijk} = \varepsilon^{ijkm}h_m, \quad \Omega_{ijk} = \varepsilon_{ijkm}h^m, \tag{5.146}$$

where ε_{ijkm} is the fully skew-symmetrical Levi-Chivita symbol.

In terms of the pseudo-vector h_m we can write the tensor (5.142) as follows

$$T_{jm} = \frac{1}{\nu}(h_j h_m - \frac{1}{2}g_{jm}h^i h_i). \tag{5.147}$$

Substituting the relationships (5.146) into (5.141), we get

$$h_{m,j} - h_{j,m} = 0. \tag{5.148}$$

These equations have two solutions: the trivial one, where $h_m = 0$, and

$$h_m = \psi_{,m}, \tag{5.149}$$

where Ψ is a pseudo-scalar.

Writing the energy-momentum tensor (5.148) in terms of this pseudo-scalar, we will have

$$T_{jm} = \frac{1}{\nu}(\psi_{,j}\psi_{,m} - \frac{1}{2}g_{jm}\psi^{,i}\psi_{,i}). \tag{5.150}$$

Tensor (5.150) is the energy-momentum tensor of a pseudo-scalar field.

Let us now decompose the Riemann tensor R_{ijkm} into irreducible parts

$$R_{ijkm} = C_{ijkm} + g_{i[k}R_{m]j} + g_{j[k}R_{m]i} + \frac{1}{3}Rg_{i[m}g_{k]j}, \tag{5.151}$$

where C_{ijkm} is the Weyl tensor; the second and third terms are the traceless part of the Ricci tensor R_{jm} and R is its trace.

Using the equations (5.133), written as

$$R_{jm} = \nu \left(T_{jm} - \frac{1}{2} g_{jm} T \right), \tag{5.152}$$

we will rewrite the relationship (5.151) as

$$R_{ijkm} = C_{ijkm} + 2\nu g_{[k(i} T_{j)m]} - \frac{1}{3} \nu T g_{i[m} g_{k]j}, \tag{5.153}$$

where T is the tensor trace (5.135).

Now we introduce the tensor current

$$J_{ijkm} = 2 g_{[k(i} T_{j)m]} - \frac{1}{3} T g_{i[m} g_{k]j} \tag{5.154}$$

and represent the tensor (5.153) as the sum

$$R_{ijkm} = C_{ijkm} + \nu J_{ijkm}. \tag{5.155}$$

Substituting this relationship into the equations (5.130), we will arrive at

$$C_{ijkm} + 2\nabla_{[k} T_{|ij|m]} + 2 T_{is[k} T^{s}_{\ |j|m]} = -\nu J_{ijkm}. \tag{5.156}$$

Equations (5.156) are the Yang-Mills equations with a geometrized source, which is defined by the relationship (5.154). In equations (5.156) for the Yang-Mills field we have the Weyl tensor C_{ijkm}, and the potentials of the Yang-Mills field are the Ricci rotation coefficients $T^{i}_{\ jk}$.

We now substitute the relationship (5.155) into the second Bianchi identities (D)

$$\nabla_{[n} R_{|ij|km]} + R^{s}_{\ j[km} T_{|is|n]} - T^{s}_{\ j[n} R_{|is|km]} = 0. \tag{5.157}$$

We thus arrive at the equations of motion

$$\nabla_{[n} C_{|ij|km]} + C^{s}_{\ j[km} T_{|is|n]} - T^{s}_{\ j[n} C_{|is|km]} = -\nu J_{nijkm} \tag{5.158}$$

for the Yang-Mills field C_{ijkm}, such that the source J_{nijkm} in them is given in terms of the current (5.154) as follows:

$$J_{nijkm} = \nabla_{[n} J_{|ij|km]} + J^{s}_{\ j[km} T_{|is|n]} - T^{s}_{\ j[n} R_{|is|km]}. \tag{5.159}$$

Using the geometrized Einstein equations (5.133) and the Yang-Mills equations (5.156), we can represent the structural Cartan equations (A) and (B) as an extended set of Einstein-Yang-Mills equations

$$\boxed{\begin{array}{cr} \nabla_{[k} e^{a}_{\ j]} + T^{i}_{\ [kj]} e^{a}_{\ i} = 0, & (A) \\ R_{jm} - \frac{1}{2} g_{jm} R = \nu T_{jm}, & (B.1) \\ C^{i}_{\ jkm} + 2\nabla_{[k} T^{i}_{\ |j|m]} + 2 T^{i}_{\ s[k} T^{s}_{\ |j|m]} = -\nu J^{i}_{\ jkm}, & (B.2) \end{array}} \tag{5.160}$$

in which the geometrized sources T_{jm} and J_{ijkm} are given by (5.135) and (5.154).

For the case of Einstein's vacuum the equations (5.160) are much simpler

$$\boxed{\begin{array}{cr} \nabla_{[k}e^a_{j]} + T^i_{[kj]}e^a{}_i = 0, & (i) \\ R_{jm} = 0, & (ii) \\ C^i{}_{jkm} + 2\nabla_{[k}T^i_{|j|m]} + 2T^i_{s[k}T^s_{|j|m]} = 0. & (iii) \end{array}} \tag{5.161}$$

The equations of motion (5.158) for the Yang-Mills field C_{ijkm} will then become

$$\nabla_{[n}C_{|ij|km]} + C^s_{j[km}T_{|is|n]} - T^s_{j[n}C_{|is|km]} = 0. \tag{5.162}$$

Equations (A) and $(B.2)$ can be written in matrix form

$$\nabla_{[k}e^a{}_{m]} - e^b{}_{[k}T^a{}_{|b|m]} = 0, \tag{A}$$

$$C^a{}_{bkm} + 2\nabla_{[k}T^a{}_{|b|m]} + 2T^a{}_{f[k}T^f{}_{|b|m]} = -\nu J^a{}_{bkm}, \tag{B.2}$$

where the current

$$J^a_{\ bkm} = 2g_{[k}{}^{(a}T_{b)m]} - \frac{1}{3}T\, g^a{}_{[m}g_{k]b}, \tag{5.163}$$

is given by

$$T^a{}_m = \frac{1}{\nu}(R^a{}_m - \frac{1}{2}g^a{}_m R), \tag{B.1}$$

$$m = 0, 1, 2, 3, \quad a = 0, 1, 2, 3.$$

By writing the equations (5.158) in matrix form, we have

$$\nabla_{[n}C^a{}_{|b|km]} + C^c{}_{b[km}T^a{}_{|c|n]} - T^c{}_{b[n}C^a{}_{|a|km]} = -\nu J^a{}_{nbkm}, \tag{5.164}$$

where

$$J^a{}_{nbkm} = \nabla_{[n}J^a{}_{|b|km]} + J^c{}_{b[km}T^a{}_{|c|n]} - T^c{}_{b[n}J^a{}_{|c|km]}. \tag{5.165}$$

Dropping the matrix indices in the matrix equations, we have

$$\nabla_{[k}e_{m]} - e_{[k}T_{m]} = 0, \tag{A}$$

$$C_{km} + 2\nabla_{[k}T_{m]} - [T_k, T_m] = -\nu J_{km}, \tag{B.2}$$

$$\nabla_n \overset{*}{C}{}^{kn} + [\overset{*}{C}{}^{kn}, T_n] = -\nu \overset{*}{J}{}^k, \tag{D}$$

where the dual matrices $\overset{*}{C}{}^{kn}$ and $\overset{*}{J}{}^k$ are given by

$$\overset{*}{C}{}^{kn} = \varepsilon^{knij}C_{ij},$$

$$\overset{*}{J}{}^{nk} = \varepsilon^{nkim}J_{im}, \tag{5.166}$$

$$\overset{*}{J}{}^k = \{\nabla_n \overset{*}{J}{}^{kn} + [\overset{*}{J}{}^{kn}, T_n]\}. \tag{5.167}$$

For the Einstein vacuum we have

$$R_{ijkm} = C_{ijkm} = \overset{*}{R}_{ijkm} = C_{ijkm}, \tag{5.168}$$

therefore the equations $(B.2)$ and (D) become simpler

$$C_{km} + 2\nabla_{[k}T_{m]} - [T_k, T_m] = 0, \tag{B.2}$$

$$\nabla_n C^{kn} + [C^{kn}, T_n] = 0. \tag{D}$$

Using the formalism of external differential forms, we can write the structural equations (A) and $(B.2)$ as follows:

$$de^a - e^b \wedge T^a_{\ b} = 0, \tag{A}$$

$$C^a_{\ b} + dT^a_{\ b} - T^a_{\ c} \wedge T^c_{\ b} = -\nu J^a_{\ b}, \tag{B.2}$$

and the equations (D) as

$$dC^a_{\ b} + C^a_{\ f} \wedge T^f_{\ b} - T^f_{\ b} \wedge C^a_{\ f} = -\nu N^a_{\ b}, \tag{D}$$

where

$$N^a_{\ b} = dJ^a_{\ b} + J^a_{\ f} \wedge T^f_{\ b} - T^f_{\ b} \wedge J^a_{\ f}. \tag{5.169}$$

Thus, the structural equations of A_4 geometry, written as (5.160), represent an extended set of Einstein-Yang-Mills equations with the gauge translations group T_4 defined on the base x^i with the structural equations (A), and with the gauge rotations group $O(3.1)$, defined in the fibre $e^i_{\ a}$ with the structural equations in the form of the geometrized equations $(B.1)$ and $(B.2)$.

5.7 Equations of geodesics of A_4 spaces

The equations of geodesics for the geometry of absolute parallelism can be obtained from the conditions of parallel vector displacement

$$u^i = \frac{dx^i}{ds} \tag{5.170}$$

with respect to the connection of A_4 geometry

$$\Delta^i_{jk} = \Gamma^i_j + T^i_{jk} = e^i_{\ a} e^a_{\ j,k}. \tag{5.171}$$

In fact, we specialize the tetrad $e^i_{\ a}$ so that the vector $e^i_{\ 0}$ would coincide with the tangent to the world line, i.e.,

$$e^i_{\ 0} = u^i = \frac{dx^i}{ds}. \tag{5.172}$$

From the relationships (5.27) for the vector (5.172) we have

$$\overset{*}{\nabla}_k u^i = u^i_{,k} + \Delta^i_{jk} u^j = 0 \tag{5.173}$$

or

$$\frac{\partial u^i}{\partial x^k} + \Gamma^i_{jk} u^j + T^i_{jk} u^j = 0. \tag{5.174}$$

Multiplying this by $u^k = dx^k/ds$ gives

$$\frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k + T^i_{jk} u^j u^k = 0 \tag{5.175}$$

or, by (5.170),

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} + T^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{5.176}$$

These four equations ($i = 0, 1, 2, 3$) are the equations of geodesics of A_4 space. They are also the equations of motion for the origin O of tetrad $e^i{}_a$. Since in the equations (5.176) the Ricci rotation coefficients T^i_{jk} have both symmetrical and skew-symmetrical parts in indices j and k

$$\begin{gathered} T^i_{jk} = T^i_{(jk)} + T^i_{[jk]} = \\ = -\Omega_{jk}^{..i} + g^{im}(g_{js}\Omega_{mk}^{..s} + g_{ks}\Omega_{mj}^{..s}), \end{gathered} \tag{5.177}$$

$$T^i_{(jk)} = g^{im}(g_{js}\Omega_{mk}^{..s} + g_{ks}\Omega_{mj}^{..s}), \tag{5.178}$$

$$T^i_{[jk]} = -\Omega_{jk}^{..i}, \tag{5.179}$$

we can write the equations (5.176) as

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} + T^i_{(jk)}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{5.180}$$

Considering the structure of the equality (5.178), we will write it in the form

$$T^i_{(jk)} = g^{im}(g_{js}\Omega_{mk}^{..s} + g_{ks}\Omega_{mj}^{..s}) = 2g^{im}\Omega_{m(jk)}, \tag{5.181}$$

hence the equations of geodesics for A_4 space can be represented as

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} + 2g^{im}\Omega_{m(jk)}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{5.182}$$

For the terns in (5.181) we can introduce the following notation:

$$\Omega_{k.j}^{.i} = g^{im} g_{ks}\Omega_{jm}^{..s}, \quad \Omega^i_{.jk} = g^{im} g_{ks}\Omega_{mj}^{..s},$$

then the contorsion tensor T^i_{jk} for space A_4 will become

$$T^i_{jk} = -\Omega_{jk}^{..i} - \Omega_{k.j}^{.i} + \Omega^i_{.jk}, \tag{5.183}$$

where

$$-\Omega_{k.j}^{.i} = \Omega^i_{.jk},$$

whence

$$T^i_{jk} = -\Omega_{jk}^{..i} + 2\Omega^i_{.jk}. \tag{5.184}$$

The covariant differential of an arbitrary vector v^i with respect to the connection (5.171) for parallel displacement from point x^i to point $x^i + dx^i$ becomes

$$\delta v^i = dv^i + \Delta^i_{jk} dx^j = 0. \tag{5.185}$$

If at an arbitrary point x^i of A_4 space we have two linear elements δx^i and dx^i and make a parallel translation of δx^i along the element dx^i, then for the final point we will have [30]

$$x^i + dx^i + \delta x^i - \Delta^i_{jk} \delta x^k dx^j = x^i + dx^i + \delta x^i + d\delta x^i. \tag{5.186}$$

On the other hand, parallel translation of the vector dx^i along the vector δx^i gives

$$x^i + \delta x^i + dx^i - \Delta^i_{jk} dx^k \delta x^j = x^i + \delta x^i + dx^i + \delta dx^i. \tag{5.187}$$

Subtracting from the relationships (5.186) the equality (5.187), we get

$$\begin{aligned} d\delta x^i - \delta dx^i = -(\Delta^i_{jk} \delta x^k dx^j + \Delta^i_{jk} dx^k \delta x^j) = \\ -(\Delta^i_{jk} - \Delta^i_{kj}) \delta x^k dx^j = -2\Delta^i_{[jk]} \delta x^k dx^j = \\ = 2\Omega_{jk}^{\cdot\cdot s} \delta x^k dx^j = -2\Omega_{jk}^{\cdot\cdot s} \delta x^j dx^k. \end{aligned} \tag{5.188}$$

Let us now consider the variation of the integral

$$\int_a^b L(x^i, u^i) ds, \tag{5.189}$$

where u^i is given by the relationship (5.170). We will write (5.188) as

$$\delta dx^i = d\delta x^i + 2\Omega_{jk}^{\cdot\cdot s} \delta x^j dx^k. \tag{5.190}$$

Then at each point of the extremum we have

$$\delta u^i = \delta \frac{dx^i}{ds} = \frac{d}{ds} \delta x^i + 2\Omega_{jk}^{\cdot\cdot i} \delta x^j \frac{dx^k}{ds}. \tag{5.191}$$

Applying a common variational procedure to the integral (5.189), we get

$$\begin{aligned} \int_a^b \delta L(x^i, u^i) ds = \\ \int_a^b \left(L(x^i + \delta x^i, u^i + \delta u^i) - L(x^i, u^i) \right) ds = \\ = \int_a^b \left(\frac{\partial L}{\partial x^i} \delta x^i + \frac{\partial L}{\partial u^i} \delta u^i \right) ds = 0. \end{aligned} \tag{5.192}$$

Substituting here the relationship (5.191) gives

$$\int_a^b \left(\frac{\partial L}{\partial x^i} \partial x^i + \frac{\partial L}{\partial u^i} \frac{d}{ds} \partial x^i + \frac{\partial L}{\partial u^i} 2\Omega_{jk}^{\cdot\cdot i} \partial x^j u^k \right) ds = 0.$$

We now integrate by parts the second term here to obtain

$$\int_a^b \left(\frac{\partial L}{\partial x^i} - \frac{d}{ds}\frac{\partial L}{\partial u^i} + 2\Omega_{ik}^{\cdot\cdot j}\frac{\partial L}{\partial u^j}u^k \right) \partial x^i = 0$$

or, since ∂x^i is arbitrary, we arrive at [30]

$$\frac{d}{ds}\frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial x^i} + 2\Omega_{ki}^{\cdot\cdot j}\frac{\partial L}{\partial u^j}u^k = 0. \tag{5.193}$$

Let now

$$L = (g_{ik}u^i u^k)^{1/2}, \tag{5.194}$$

along the extremum $L = 1$ by the relationship

$$g_{ik}u^i u^k = u^i u_i = 1.$$

Substituting the Lagrangian (5.194) into equations (5.193) gives

$$g_{mi}\frac{du^i}{ds} + \Gamma_{mjk}u^j u^k + 2\Omega_{mj}^{\cdot\cdot s}g_{sk}u^k u^j = 0. \tag{5.195}$$

Multiplying this relationship by g^{im}, we get

$$\frac{du^i}{ds} + \Gamma^i_{kj}u^j u^k + 2g^{im}g_{ks}\Omega_{mj}^{\cdot\cdot s}u^j u^k = 0$$

or

$$\frac{du^i}{ds} + \Gamma^i_{kj}u^j u^k + 2g^{im}\Omega_{m(jk)}u^j u^k = 0. \tag{5.196}$$

We have thus obtained, using the variational principle, the equations of the geodesics in the form (5.182). Consider now the equations that describe the variation of the orientation of the tetrad $e^i{}_a$ as it moves according to the equations of the geodesics (5.196). We will rewrite the equations (5.43) as

$$\partial_k e^i{}_a + \Delta^i_{jk}e^j{}_a = 0$$

or

$$de^i{}_a + \Delta^i_{jk}e^j{}_a dx^k = 0. \tag{5.197}$$

Dividing these equations by ds yields

$$\frac{de^i{}_a}{ds} + \Delta^i_{jk}e^j{}_a\frac{dx^k}{ds} = 0. \tag{5.198}$$

Further, taking the second derivative $d^2e^i{}_a/ds^2$, we will have

$$\frac{d}{ds}\left(\frac{de^i{}_a}{ds}\right) = \frac{d}{ds}\left(\frac{\partial e^i{}_a}{\partial x^k}\frac{dx^k}{ds}\right) = \frac{\partial^2 e^i{}_a}{\partial x^m \partial x^k}\frac{dx^k}{ds}\frac{dx^m}{ds} + \frac{\partial e^i{}_a}{\partial x^k}\frac{d^2x^k}{ds^2}. \tag{5.199}$$

Since

$$\frac{\partial^2 e^i{}_a}{\partial x^m \partial x^k} = \frac{\partial}{\partial x^m}(-\Delta^i_{ak}) = \Delta^i_{jk,m} e^j{}_a -$$
$$-\Delta^i_{sk}(-\Delta^s_{jm} e^j{}_a) = (-\Delta^i_{jk,m} + \Delta^i_{sk}\Delta^s_{jm}) e^j{}_a$$

and

$$\frac{\partial e^i{}_a}{\partial x^k}\frac{d^2 x^k}{ds^2} = \Delta^i_{js}\Delta^s_{km}\frac{dx^k}{ds}\frac{dx^m}{ds}e^j{}_a,$$

we have

$$\frac{d^2 e^i{}_a}{ds^2} + (\Delta^i_{jk,m} - \Delta^i_{sk}\Delta^s_{jm} - \Delta^i_{js}\Delta^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds}e^j{}_a = 0. \qquad (5.200)$$

Substituting here the sum (5.171), we have

$$\frac{d^2 e^i{}_a}{ds^2} + (\Gamma^i_{jk,m} + T^i_{jk,m} - \Gamma^i_{sk}\Gamma^s_{jm} - \Gamma^i_{sk}T^s_{jm} -$$
$$-T^i_{sk}\Gamma^s_{jm} - T^i_{sk}T^s_{jm} - \Gamma^i_{js}\Gamma^s_{km} - T^i_{js}\Gamma^s_{km} -$$
$$-\Gamma^i_{js}T^s_{km} - T^i_{js}T^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds}e^j{}_a = 0. \qquad (5.201)$$

Since independent equations (5.201) (for three Euler's angles and three pseudo-Euclidean angles) describe the variation of the orientation of tetrad $e^i{}_a$ as it moves from the origin O according to the equations of geodesics (5.196).

In A_4 spaces, where the metric is flat

$$g_{ik} = \eta_{ik} = \mathrm{diag}(1 - 1 - 1 - 1), \qquad (5.202)$$

the Christoffel symbols $\Gamma^i{}_{js}$ vanish and the equations (5.201) become

$$\frac{d^2 e^i{}_a}{ds^2} + (T^i_{jk,m} - T^i_{sk}T^s_{jm} - T^i_{js}T^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds}e^j{}_a = 0, \qquad (5.203)$$

and the equations of geodesics (5.175) will become

$$\frac{d^2 x^i}{ds^2} + T^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \qquad (5.204)$$

We now introduce the tensor of the four-dimensional angular velocity of rotation tetrads $e^i{}_a$ [33]

$$\Omega_{ij} = T_{ijk}\frac{dx^k}{ds} = -\frac{de_{ia}}{ds}e^a{}_j = \frac{de_{ja}}{ds}e^a{}_i \qquad (5.205)$$

with the symmetry properties

$$\Omega_{ij} = -\Omega_{ji}, \qquad (5.206)$$

determined by the symmetry (5.48), for which the Ricci rotation coefficients hold.

Using (5.205), we will write the equations (5.204) and (5.203) as

$$\frac{d^2x^i}{ds^2}+\Omega^i{}_j\frac{dx^j}{ds}=0, \tag{5.207}$$

$$\frac{d\Omega^i{}_j}{ds}-T^i{}_{jk,m}\frac{dx^k}{ds}\frac{dx^m}{ds}+T^i{}_{js}\Omega^s{}_k\frac{dx_k}{ds}=0. \tag{5.208}$$

The skew-symmetric matrix (5.206) can be represented as

$$\Omega_{ij}=\begin{pmatrix} 0 & \Omega_{01} & \Omega_{02} & \Omega_{03} \\ \Omega_{10} & 0 & \Omega_{12} & \Omega_{13} \\ \Omega_{20} & \Omega_{21} & 0 & \Omega_{23} \\ \Omega_{30} & \Omega_{31} & \Omega_{32} & 0 \end{pmatrix} \tag{5.209}$$

Let us now give a physical interpretation of the components of the matrix (5.209). We multiply the equations (5.207) by the mass m and rewrite them as

$$m\frac{d^2x_i}{ds^2}+m\Omega_{ij}\frac{dx^j}{ds}=0. \tag{5.210}$$

If the condition (5.202) holds, there equations can be represented as

$$m\frac{du_i}{ds_o}+m\Omega_{ij}\frac{dx^j}{ds_o}=0, \tag{5.211}$$

where

$$ds_o=(\eta_{ik}dx^idx^k)^{1/2} \tag{5.212}$$

is the pseudo-Euclidean metric and $u_i=dx_i/ds_o$.

We represent the equations (5.211) in the form

$$m\frac{du_i}{ds_o}=-mT_{i(jk)}\frac{dx^j}{ds_o}\frac{dx^k}{ds_o}, \tag{5.213}$$

where the part of T symmetric in indices j and k is given by (5.178).

Assuming that motion governed by the equations (5.213) is nonrelativistic ($v/c \ll 1$), we will write the three-dimensional part of these equations as

$$m\frac{du_\alpha}{ds_o}=-mT_{\alpha(ok)}\frac{dx^o}{ds_o}\frac{dx^k}{ds_o}-2mT_{\alpha(\beta k)}\frac{dx^\beta}{ds_o}\frac{dx^k}{ds_o} \tag{5.214}$$

or, from the relationship (5.205), as

$$m\frac{du_\alpha}{ds_o}=-m\Omega_{\alpha o}\frac{dx^o}{ds_o}-2m\Omega_{\alpha\beta}\frac{dx^\beta}{ds_o}. \tag{5.215}$$

Since in the nonrelativistic approximation

$$ds_o=cdt, \quad u_a=\frac{v_\alpha}{c},$$

and $dx_o = cdt$, the equations (5.215) can be written as

$$m\frac{dv_\alpha}{dt} = -mc^2\Omega_{\alpha o} - 2mc^2\Omega_{\alpha\beta}\frac{1}{c}\frac{dx^\beta}{dt}. \qquad (5.216)$$

It is known from classical mechanics that the nonrelativistic equations of motion of the origin O of a three-dimensional accelerated reference frame under inertia forces alone have the form [34]

$$\frac{d}{dt}(m\mathbf{v}) = m\left(-\mathbf{W} + 2[\mathbf{v}\omega]\right), \qquad (5.217)$$

where $\mathbf{W}$ is the vector of translational acceleration, and ω is the vector of the three-dimensional angular velocity of rotation of the accelerated reference frame.

We write these equations as

$$\frac{d}{dt}(mv_\alpha) = m\left(-W_{\alpha o} + 2\omega_{\alpha\beta}\frac{dx^\beta}{dt}\right), \qquad (5.218)$$

where $\mathbf{W} = (W_{10}, W_{20}, W_{30})$,

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} = -\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad (5.219)$$

$$\omega = (\omega_1, \omega_2, \omega_3),$$

and comparing these with (5.217), we obtain

$$\Omega_{10} = \frac{W_1}{c^2}, \quad \Omega_{20} = \frac{W_2}{c^2}, \quad \Omega_{30} = \frac{W_3}{c^2},$$

$$\Omega_{12} = -\frac{\omega_3}{c}, \quad \Omega_{13} = \frac{\omega_2}{c}, \quad \Omega_{23} = -\frac{\omega_1}{c}.$$

Therefore, the matrix (5.209) in this case has the form

$$\Omega_{ij} = \frac{1}{c^2}\begin{pmatrix} 0 & -W_1 & -W_2 & -W_3 \\ W_1 & 0 & -c\omega_3 & c\omega_2 \\ W_2 & c\omega_3 & 0 & -c\omega_1 \\ W_3 & -c\omega_2 & c\omega_1 & 0 \end{pmatrix} \qquad (5.220)$$

It is seen from this matrix that the four-dimensional rotation of the tetrad $e^i{}_a$, caused by the torsion of the A_4 spaces, gives rise in physics to inertia fields associated with translational and rotational accelerations.

5.8 Structural equations of right and left A_4 geometry

We can consider three forms of the geometry of absolute parallelism.

(1) A_4 geometry, with the nonzero Riemannian tensor $R^i{}_{jkm}$ and torsion $\Omega^{..i}_{jk}$. The structural Cartan equations then become

$$\nabla_{[k}e^a{}_{j]} + T^i{}_{[kj]}e^a{}_i = 0, \tag{5.221}$$

$$R^i{}_{jkm} + 2\nabla_{[k}T^i_{|j|m]} + 2T^i{}_{s[k}T^s_{|j|m]} = 0. \tag{5.222}$$

(2) A_4 geometry, with the zero Riemannian $R^i{}_{jkm}$ and nonzero torsion $\Omega^{..i}_{jk}$. In that case the structural Cartan equations can be written as

$$\nabla_{[k}e^a_{j]} + T^i_{[kj]}e^a{}_i = 0, \tag{5.223}$$

$$\nabla_{[k}T^i_{|j|m]} + T^i_{s[k}T^s_{|j|m]} = 0. \tag{5.224}$$

(3) A_4 geometry, with the zero Riemannian tensor $R^i{}_{jkm}$ and noncoordinate torsion $\Omega^{..i}_{jk}$. The structural Cartan equations of the geometry coincide with the structural equations of the pseudo-Euclidean space E_4, and they look like

$$\overset{\circ}{\nabla}_{[k}\overset{\circ}{e}{}^a{}_{j]} + \overset{\circ}{T}{}^i{}_{[kj]}\,\overset{\circ}{e}{}^a{}_i = 0, \tag{5.225}$$

$$\overset{\circ}{\nabla}_{[k}\overset{\circ}{T}{}^i{}_{|j|m]} + \overset{\circ}{T}{}^i{}_{s[k}\,\overset{\circ}{T}{}^s{}_{|j|m]} = 0, \tag{5.226}$$

where the tetrad $\overset{\circ}{e}{}^a_i$ determines the "coordinate torsion"

$$\overset{\circ}{\Omega}{}^{..i}_{jk} = \overset{\circ}{e}{}^i_a\,\overset{\circ}{e}{}^a{}_{[k,j]} = \frac{1}{2}\,\overset{\circ}{e}{}^i{}_a(\overset{\circ}{e}{}^a{}_{k,j} - \overset{\circ}{e}{}^a{}_{j,k}). \tag{5.227}$$

Since in the pseudo-Euclidean space the T_4 and $O(3.1)$ groups hold globally and its internal geometry is trivial, then, for example, in the Cartesian coordinate $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ the structural equations (5.225) and (5.226) become the identities

$$0 \equiv 0, \tag{5.228}$$

$$0 \equiv 0. \tag{5.229}$$

If we now go over to the spherical coordinates

$$x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi,$$

we will get the equations (5.225)–(5.227), which include:
(a) components of the "coordinate" tetrad

$$\overset{\circ}{e}{}^{(0)}_0 = \overset{\circ}{e}{}^{(1)}_1 = 1, \quad \overset{\circ}{e}{}^{(2)}_2 = r, \quad \overset{\circ}{e}{}^{(3)}_3 = r\sin\theta,$$
$$\overset{\circ}{e}{}^0_{(0)} = \overset{\circ}{e}{}^1_{(1)} = 1, \quad \overset{\circ}{e}{}^2_{(2)} = \frac{1}{r}, \quad \overset{\circ}{e}{}^3_{(3)} = \frac{1}{r\sin\theta}; \tag{5.230}$$

(b) components of "coordinate torsion"

$$\overset{\circ}{\Omega}{}_{21}^{..2} = \overset{\circ}{\Omega}{}_{31}^{..3} = -(2r)^{-1}, \quad \overset{\circ}{\Omega}{}_{32}^{..3} = -\frac{1}{2}\cot\theta; \tag{5.231}$$

(c) components of the Ricci rotation coefficients

$$\begin{gathered} \overset{\circ}{T}{}^{1}{}_{22} = r, \quad \overset{\circ}{T}{}^{1}{}_{33} = r\sin^2\theta, \quad \overset{\circ}{T}{}^{2}{}_{33} = \sin\theta\cos\theta, \\ \overset{\circ}{T}{}^{2}{}_{12} = \overset{\circ}{T}{}^{3}{}_{13} = -\frac{1}{r}, \quad \overset{\circ}{T}{}^{3}{}_{23} = -\cot\theta. \end{gathered} \tag{5.232}$$

Using the formulas

$$\overset{\circ}{g}_{ik} = \eta_{ab} \overset{\circ}{e}{}^{a}_{i} \overset{\circ}{e}{}^{b}_{k}, \quad \eta_{ab} = \eta^{ab} = \mathrm{diag}(1\ -1\ -1\ -1),$$

we find the components of the metric tensor

$$\overset{\circ}{g}_{00} = \overset{\circ}{g}_{11} = 1, \quad \overset{\circ}{g}_{22} = -r, \quad \overset{\circ}{g}_{33} = -r^2\sin^2\theta,$$

the metric

$$ds_o^2 = \overset{\circ}{g}_{ij}\, dx^i dx^j = c^2dt^2 - dr^2 - r^2(de^2 + \sin^2\theta d\varphi^2)$$

and the components of the Christoffel symbols

$$\begin{gathered} \overset{\circ}{\Gamma}{}^{1}{}_{33} = -r\sin^2\theta, \quad \overset{\circ}{\Gamma}{}^{1}{}_{22} = -r, \quad \overset{\circ}{\Gamma}{}^{2}{}_{12} = \overset{\circ}{\Gamma}{}^{3}{}_{13} = \tfrac{1}{r}, \\ \overset{\circ}{\Gamma}{}^{2}{}_{33} = -\sin\theta\cos\theta, \quad \overset{\circ}{\Gamma}{}^{3}{}_{23} = \cot\theta. \end{gathered} \tag{5.233}$$

Thus, in the pseudo-Euclidean geometry A_4, when we deviate from Cartesian coordinates, instead of the identities (5.228) and (5.229) we get the "coordinate structural equations" (5.225) and (5.226).

Suppose now that the initial pseudo-Euclidean space A_4 is deformed in a continuous manner (e.g., using conformal transformations) into an A_4 space with a nonzero dynamic torsion field and the structural equations (5.223) and (5.224). We can distinguish the right

$$\overset{+}{\Omega}{}_{jk}^{..i} = r^i{}_a r^a{}_{[k,j]} = \frac{1}{2} r^i{}_a (r^a{}_{k,j} - r^a{}_{j,k}) \tag{5.234}$$

and left

$$\overset{-}{\Omega}{}_{jk}^{..i} = l^i{}_a l^a{}_{[k,j]} = \frac{1}{2} l^i{}_a (l^a{}_{k,j} - l^a{}_{j,k}) \tag{5.235}$$

torsion fields. In these equations r^i_a and l^i_a stand for the right and left tetrads. respectively.

We well take the right tetrad r^i_a to mean a tetrad $\overset{+}{e}{}^{i}{}_{a}$, such that when the three-dimensional spatial part rotates from the x axis to the y axis the vector of the angular rotational velocity points along the z axis, so that the rotation occurs counterclockwise if looking from the side to which the z vector points.

For example, the four-dimensional rotation matrix (5.220) for the right tetrad looks like

$$\overset{+}{\Omega}_{ij}=\frac{1}{c^2}\begin{pmatrix} 0 & -W_1 & -W_2 & -W_3 \\ W_1 & 0 & -c\omega_3 & c\omega_2 \\ W_2 & c\omega_3 & 0 & -c\omega_1 \\ W_3 & -c\omega_2 & c\omega_1 & 0 \end{pmatrix} \tag{5.236}$$

whereas for the left rotations we have

$$\overline{\Omega}_{ij}=\frac{1}{c^2}\begin{pmatrix} 0 & W_1 & W_2 & W_3 \\ -W_1 & 0 & c\omega_3 & -c\omega_2 \\ -W_2 & -c\omega_3 & 0 & c\omega_1 \\ -W_3 & c\omega_2 & -c\omega_1 & 0 \end{pmatrix}. \tag{5.237}$$

It is seen that

$$\overset{+}{\Omega}_{ij}= - \overset{-}{\Omega}_{ij} . \tag{5.238}$$

From (5.205) and (5.238), we have

$$\overset{+}{T}{}^{i}_{\ jk} = - \overset{-}{T}{}^{i}_{\ jk}. \tag{5.239}$$

Since the metric tensor g_{ik} is determined both by the right and left tetrad in a similar manner [35]

$$g_{ik} = \eta_{ab} r^a_i r^b_k = \eta_{ab} l^a_i l^b_k, \tag{5.240}$$

it follows from the definition

$$T^i_{jk} = -\Omega^{..i}_{jk} + g^{im}(g_{js}\Omega^{..s}_{mk} + g_{ks}\Omega^{..s}_{mj}) \tag{5.241}$$

that the components (5.234) and (5.235) of the right and left torsion fields differ in sign

$$\overset{+}{\Omega}{}^{..i}_{jk} = - \overset{-}{\Omega}{}^{..i}_{jk}. \tag{5.242}$$

By dividing the torsion fields into left- and right-hand ones, we thereby split the translations group T_4 into the right $\overset{+}{T}_4$ and left $\overset{-}{T}_4$ translations groups; and the rotations group $O(3.1)$ into the right $SO^+(3.1)$ and left $SO^-(3.1)$ rotations group.

We will write the structural Cartan equations of the A_4 geometry, which are transformed using continuous transformations in T_4 and $SO^+(3.1)$ groups, as follows:

$$\nabla_{[k} \overset{+}{e}{}^{a}_{\ j]} + \overset{+}{T}{}^{i}_{\ [kj]} \overset{+}{e}{}^{a}_{\ i} = 0, \tag{5.243}$$

$$\nabla_{[k} \overset{+}{T}{}^{i}_{\ |j|m]} + \overset{+}{T}{}^{i}_{\ s[k} \overset{+}{T}{}^{s}_{\ |j|m]} = 0. \tag{5.244}$$

Accordingly, the equations

$$\nabla_{[k}\overline{e}^a_{j]} + \overline{T}^i_{[kj]}\overline{e}^a_i = 0, \tag{5.245}$$

$$\nabla_{[k}\overline{T}^{i}_{\ |j|m]} + \overline{T}^{i}_{\ s[k}\overline{T}^{s}_{\ |j|m]} = 0 \qquad (5.246)$$

are transformed continuously in the T_4 and $SO^+(3.1)$ groups.

It is clear that discrete transformations — inversion transformations — enable us to transform the right equations (5.243) and (5.244) into left equations (5.245) and (5.246), and vice versa.

The property (5.242) of the A_4 geometry enables an empty pseudo-Euclidean geometry to be "split" into right- and left-hand geometries:

$$\overset{\circ}{\Omega}{}_{jk}^{\cdot\cdot i} = \overset{+}{\Omega}{}_{jk}^{\cdot\cdot i} + \overline{\Omega}{}_{jk}^{\cdot\cdot i} = 0, \qquad (5.247)$$

whose torsion is nonzero. This property appeared to be quite useful for the description of the production of matter from "nothing" in the theory of physical vacuum [36].

If now we split the structural Cartan equations (5.221) and (5.222) into right and left ones, we will get

$$\nabla_{[k}\overset{+}{e}{}^{a}_{\ j]} + \overset{+}{T}{}^{i}_{\ [kj]}\overset{+}{e}{}^{a}_{\ i} = 0, \qquad (5.248)$$

$$\overset{+}{R}{}^{i}_{\ jkm} + 2\nabla_{[k}\overset{+}{T}{}^{i}_{\ |j|m]} + 2\overset{+}{T}{}^{i}_{\ s[k}\overset{+}{T}{}^{s}_{\ |j|m]} = 0, \qquad (5.249)$$

$$\nabla_{[k}\overline{e}{}^{a}_{\ j]} + \overline{T}{}^{i}_{\ [kj]}\overline{e}{}^{a}_{\ i} = 0, \qquad (5.250)$$

$$\overline{R}{}^{i}_{\ jkm} + 2\nabla_{[k}\overline{T}{}^{i}_{\ |j|m]} + 2\overline{T}{}^{i}_{\ s[k}\overline{T}{}^{s}_{\ |j|m]} = 0. \qquad (5.251)$$

Writing the structural Cartan equations as the extended right and left Einstein-Yang-Mills equations, we will arrive at

$$\boxed{\begin{array}{lr} \nabla_{[k}\overset{+}{e}{}^{a}_{\ j]} + \overset{+}{T}{}^{i}_{\ [kj]}\overset{+}{e}{}^{a}_{\ i} = 0, & (\overset{+}{A}) \\ \overset{+}{R}_{jm} - \frac{1}{2}g_{jm}\overset{+}{R} = \nu\overset{+}{T}_{jm}, & (\overset{+}{B}.1) \\ \overset{+}{C}{}^{i}_{\ jkm} + 2\nabla_{[k}\overset{+}{T}{}^{i}_{\ |j|m]} + 2\overset{+}{T}{}^{i}_{\ s[k}\overset{+}{T}{}^{s}_{\ |j|m]} = -\nu\overset{+}{J}{}^{i}_{\ jkm}. & (\overset{+}{B}.2) \end{array}} \qquad (5.252)$$

$$\boxed{\begin{array}{lr} \nabla_{[k}\overline{e}{}^{a}_{\ j]} + \overline{T}{}^{i}_{\ [kj]}\overline{e}{}^{a}_{\ i} = 0, & (\overline{A}) \\ \overline{R}_{jm} - \frac{1}{2}g_{jm}\overline{R} = \nu\overline{T}_{jm}, & (\overline{B}.1) \\ \overline{C}{}^{i}_{\ jkm} + 2\nabla_{[k}\overline{T}{}^{i}_{\ |j|m]} + 2\overline{T}{}^{i}_{\ s[k}\overline{T}{}^{s}_{\ |j|m]} = -\nu\overline{J}{}^{i}_{\ jkm}. & (\overline{B}.2) \end{array}} \qquad (5.253)$$

In the theory of physical vacuum that is based on the universal relativity principle [37], equations (5.252) and (5.253) describe the right and left matter produced from vacuum.

Chapter 6

The geometry of absolute parallelism in spinor basis

6.1 Three main spinor bases of A_4 geometry

The geometry of absolute parallelism, as laid down in vector basis, enables the structural equations of this geometry of be represented as right (invariant with respect to the T_4^+ and $SO^+(3.1)$) groups and left (invariant with respect to the T_4^- and the $SO^-(3.1)$ groups) groups of the structural equations (A^+), (B^+) and (A^-) and (B^-), respectively. Equations (A^+), (B^+) (or (A^-), (B^-)) can, in turn, be split by a transition into a group of equations, whose component fields have opposite spins. For this purpose, we have to use spinor basis and some elements of spinor analysis.

We will view the spinor geometry A_4 as a differentiable manifold X_4, such that at each point M with the translational coordinates x $(i = 0, 1, 2, 3)$ a two-dimensional spinor space $\mathcal{C}^2$ is introduced [38]. There are three possibilities for introducing the spinor basis in the spinor space $\mathcal{C}^2$:

(a) spinor Γ-basis formed by the Infeld-Van der Werden symbols $\sigma^i_{\alpha\dot\beta}$ [39], which satisfy the equality

$$\nabla_n \sigma^i_{\alpha\dot\beta} = 0; \tag{6.1}$$

(b) spinor Δ-basis formed by the Newman-Penrose symbols $\sigma^i_{A\dot B}$ [40], which satisfy the equality

$$\overset{*}{\nabla}_n \sigma^i_{A\dot B} = 0; \tag{6.2}$$

(c) spinor dyad basis ξ^α_B, which satisfies the equality [41]

$$\varepsilon^{BD}\xi_{\alpha D}\nabla_k \xi^\alpha_B = 0. \tag{6.3}$$

In relationships (6.1)–(6.3) the indices $\alpha, \dot\beta, \ldots$ and $A, \dot B, \ldots$ are spinor indices that take on the values 0,1 and $\dot 0, \dot 1$. Any local vector A^i that belongs to

$\mathcal{C}^2$ can be represented as a spin-tensor of the second rank either in the spinor Γ- basis

$$A^i = A^{\alpha\dot\beta}\sigma^i_{\alpha\dot\beta}, \tag{6.4}$$

or in the spinor Δ-basis

$$A^i = A^{A\dot B}\sigma^i_{A\dot B}. \tag{6.5}$$

All the spin-tensors associated with the Γ-basis will have the spinor indices $\alpha, \dot\beta, \ldots$, and the spin-tensors associated with Δ basis will have spinor indices $A, \dot B, \ldots$. As to dyad ξ^α_B, it is a connection between Γ- and Δ-basis

$$\sigma^i_{A\dot B} = \sigma^i_{\alpha\dot\beta}\xi^\alpha_A\overline{\xi}^{\dot\beta}_B. \tag{6.6}$$

Here

$$\overline{\xi}^{\dot\beta}_B = \overline{\xi^\beta_B},$$

and the bar on the right-hand side of the equality implies complex conjugation.

Spinor Δ-basis is connected with the vector basis e^a_i by

$$\sigma^i_{A\dot B} = e^i_a\sigma^a_{A\dot B}, \tag{6.7}$$

$$\sigma^{A\dot B}_i = e^a_i\sigma^{A\dot B}_a, \tag{6.8}$$

where $\sigma_i{}^{A\dot B}$ are complex Hermitian ($\overline{\sigma_i{}^{A\dot B}} = \sigma_i{}^{\dot A B}$) matrices, and the matrices $\sigma^a{}_{A\dot B}$ and $\sigma_a{}^{A\dot B}$ have the form

$$\sigma^a{}_{A\dot B} = (2)^{-1/2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{6.9}$$

$$\sigma^{A\dot B}_a = (2)^{-1/2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{6.10}$$

where

$$\det(\sigma^a_{A\dot B}) = i, \quad \det(\sigma^{A\dot B}_a) = -i.$$

From the orthogonality conditions for the tetrad e^i_a

$$e^a{}_i e^j{}_a = \delta_i{}^j, \quad e^a{}_i e^i{}_b = \delta^a_b \tag{6.11}$$

and the relationships (6.7)–(6.10) follows the orthogonality conditions for the spinor Δ-basis

$$\sigma^{A\dot B}_i\sigma^j_{A\dot B} = \delta_i{}^j, \tag{6.12}$$

$$\sigma^{A\dot B}_i\sigma^i_{C\dot E} = \delta^A{}_C\delta^{\dot B}{}_{\dot E}. \tag{6.13}$$

For the spinor Γ-basis the following orthogonality conditions hold [54]

$$\sigma_i^{\alpha\dot{\beta}}\sigma^j{}_{\alpha\dot{\beta}} = \delta_i{}^j, \tag{6.14}$$

$$\sigma_i^{\alpha\dot{\beta}}\sigma^i_{\rho\dot{\nu}} = \delta^\alpha{}_\rho\delta^{\dot{\beta}}{}_{\dot{\nu}}. \tag{6.15}$$

Whence, by (6.6) and (6.12)-(6.13), follow the orthogonality conditions for the spinor dyad

$$\begin{gathered} \xi^o_\alpha\xi^\alpha_1 = 1, \\ \xi^\alpha_o\xi^o_\alpha = -\xi^o_\alpha\xi^\alpha_o = 0, \\ \xi^\alpha_1\xi^1_\alpha = 0. \end{gathered} \tag{6.16}$$

In addition, there are the relationships [54]

$$\begin{gathered} \xi^o_\alpha\xi^\beta_o - \xi^1_\alpha\xi^\beta_o = \delta^\beta_\alpha, \\ \xi^o_\alpha\xi^1_\beta - \xi^1_\alpha\xi^o_\beta = \varepsilon_{\alpha\beta}, \end{gathered} \tag{6.17}$$

where

$$\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = \varepsilon_{\dot{\gamma}\dot{\delta}} = \varepsilon^{\dot{\gamma}\dot{\delta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.18}$$

is the fundamental spinor [40] that obeys the following relationships:

$$\varepsilon_{\alpha\beta}\varepsilon^{\kappa\beta} = \varepsilon^\kappa_\alpha = -\varepsilon^\kappa_\alpha, \tag{6.19}$$

$$\varepsilon_{\alpha\beta}\varepsilon^{\kappa\pi} = \delta^\kappa_\alpha\delta^\pi_\beta - \delta^\pi_\alpha\delta^\kappa_\beta, \tag{6.20}$$

$$\varepsilon^\alpha_\alpha = 2, \tag{6.21}$$

$$\varepsilon_{\alpha[\beta}\varepsilon_{\kappa\delta]} = 0, \tag{6.22}$$

$$\varepsilon^\beta_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.23}$$

The fundamental spinor $\varepsilon_{\alpha\beta}$ increases and decreases the indices on the spin-tensor associated with the Γ-basis, similar to the metric tensor g_{ik} in the vector basis. In the spinor Δ-basis it has the form

$$\varepsilon_{AB} = \varepsilon_{\alpha\beta}\xi^\alpha_A\xi^\beta_B, \tag{6.24}$$

so that

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon^{\dot{C}\dot{D}} = \varepsilon_{\dot{C}\dot{D}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.25}$$

The fundamental spinor ε_{AB} increases and decreases indices on the spin-tensors associated with the Δ-basis. For example, we have

$$\begin{aligned} &\chi^{\dots A\dots}_{\dots\dots\dots}\varepsilon_{AB} = \chi^{\dots\dots\dots}_{\dots B\dots}, \quad \varepsilon^{AB}\chi^{\dots\dots\dots}_{\dots B\dots} = \chi^{\dots A\dots}_{\dots\dots\dots}, \\ &\varphi^{\dots\dot{A}\dots}_{\dots\dots}\varepsilon_{\dot{A}\dot{B}} = \varphi^{\dots\dots\dots}_{\dots\dot{B}\dots} \quad \varepsilon^{\dot{A}\dot{B}}\varphi^{\dots\dots\dots}_{\dots\dot{B}\dots} = \varphi^{\dots\dot{A}\dots}_{\dots\dots}. \end{aligned} \tag{6.26}$$

If the spinor is skew-symmetric in two indices

$$\theta_{\dots A\dots B\dots} = -\theta_{\dots B\dots A\dots}, \tag{6.27}$$

then, using the fundamental spinor ε_{AB}, it can be represented as [40]

$$\theta_{\dots A\dots B\dots} = \frac{1}{2}\varepsilon_{AB}\theta_{\dots C\dots}^{\dots\dots C\dots}. \tag{6.28}$$

The same properties are valid in the spinor Γ-basis for the fundamental spinor $\varepsilon_{\alpha\beta}$.

6.2 Spinor representation of the structural Cartan equations of A_4 geometry

The relationship (6.28) makes it possible to reduce spinors skew-symmetric in primed and unprimed indices to spinors that are completely (or partially) symmetrical in primed and anprimed indices. In the space of spinors of this type irreducible representations of the groups $SL(2.C)$ are realized [42]. This group replaces the group $SO(3.1)$ on passing over to the spinor basis.

Definition 6.1. We will say that the components of a spinor with r symmetrical lower indices and with s symmetrical lower primed indices are transformed in $D(r/2, s/2)$ irreducible representation of the group $SL(2.C)$.

For example, the spinor

$$F_{AB} = F_{BA}$$

is transformed in $D(1.0)$, and the spinor

$$F_{\dot{C}\dot{D}} = F_{\dot{D}\dot{C}}$$

in the $D(0.1)$ irreducible representation of the group $SL(2.C)$.

We will write the main relationships of the A_4 geometry in the spinor Δ-basis. This can be accomplished using the spinor representation of the arbitrary tensor $T_{\dots\dots j\dots}^{\dots i\dots\dots}$ in the Δ-basis

$$T_{\dots\dots C\dot{E}\dots}^{\dots A\dot{B}\dots\dots} = \sigma_i^{A\dot{B}} T_{\dots\dots j\dots}^{\dots i\dots\dots}\sigma^j_{C\dot{E}} \tag{6.29}$$

or simply replacing the matrix indices by two spinor ones as follows:

$$e_i^a \leftrightarrow \sigma_i^{A\dot{B}}, \tag{6.30}$$

$$T^a{}_{bm} \leftrightarrow T^{A\dot{B}}{}_{C\dot{D}m}, \tag{6.31}$$

$$R^a{}_{bkm} \leftrightarrow R^{A\dot{B}}{}_{C\dot{D}km}, \tag{6.32}$$

$$\eta_{ab} \leftrightarrow \eta_{A\dot{B}C\dot{D}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}, \tag{6.33}$$

and so on.

Proposition 6.1. In the spinor Δ-basis the metric tensor g_{ij} of the A_4 geometry has the form

$$g_{ij} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\sigma_i^{A\dot{B}}\sigma_j^{C\dot{D}}. \tag{6.34}$$

Proof. Substituting into

$$g_{ij} = \eta_{ab}e^a{}_i e^b{}_j$$

the relationships (6.7) and (6.8) written as

$$e^a{}_i = \sigma_i^{A\dot{B}}\sigma^a{}_{A\dot{B}}, \quad e^b{}_j = \sigma^{C\dot{D}}{}_j\sigma^b{}_{C\dot{D}}, \tag{6.35}$$

we have

$$g_{ij} = \eta_{ab}\sigma_i^{A\dot{B}}\sigma^a_{A\dot{B}}\sigma_j^{C\dot{D}}\sigma^b_{C\dot{D}}. \tag{6.36}$$

From the relationships (6.9), (6.10), (6.25) and the definition

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1\ -1\ -1\ -1),$$

we obtain the following equality:

$$\eta_{ab}\sigma^a_{A\dot{B}}\sigma^b_{C\dot{D}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}.$$

Substituting this into (6.36), we arrive at the formula (6.34).

We now write the structural Cartan equations in matrix form

$$\nabla_{[k}e^a_{m]} - e^b_{[k}T^a_{|b|m]} = 0, \tag{A}$$

$$R^a_{bkm} + 2\nabla_{[k}T^a_{|b|m]} + 2T^a_{c[k}T^c_{|b|m]} = 0. \tag{B}$$

Using the rules (6.30)-(6.32), we write these equations in the spinor Δ-basis

$$\nabla_{[k}\sigma^{A\dot{B}}_{m]} - \sigma^{C\dot{D}}_{[k}T^{A\dot{B}}{}_{|C\dot{D}|m]} = 0, \tag{6.37}$$

$$R^{A\dot{B}}{}_{C\dot{D}km} + 2\nabla_{[k}T^{A\dot{B}}{}_{|C\dot{D}|m]} + 2T^{A\dot{B}}{}_{E\dot{F}[k}T^{E\dot{F}}{}_{|C\dot{D}|m]} = 0. \tag{6.38}$$

Consequently, the second Bianchi identity of the A_4 geometry

$$\nabla_{[n}R^a_{|b|km]} + R^c_{b[km}T^a_{|c|n]} - T^c_{b[n}R^a_{|c|km]} = 0 \tag{D}$$

in the spinor Δ-basis becomes

$$\nabla_{[n}R^{A\dot{B}}{}_{|C\dot{D}|km]} + R^{E\dot{F}}{}_{C\dot{D}[km}T^{A\dot{B}}{}_{|C\dot{D}|n]} - T^{E\dot{F}}{}_{C\dot{D}[n}R^{A\dot{E}}{}_{|E\dot{F}|km]} = 0. \tag{6.39}$$

Proposition 6.2. If $F_{ij} = -F_{ji}$ is a real skew-symmetrical tensor, then the corresponding spinor

$$F_{A\dot{B}C\dot{D}} = F_{ij}\sigma^i{}_{A\dot{B}}\sigma^j_{C\dot{D}} \tag{6.40}$$

can be represented in the form

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}F_{AC} + \varepsilon_{AC}\overline{F}_{\dot{B}\dot{D}}), \tag{6.41}$$

where the spinor

$$F_{AC} = F_{CA} \tag{6.42}$$

is transformed in the $D(1.0)$ irreducible representation of the group $SL(2.C)$, and the spinor

$$\overline{F}_{BD} = F_{\dot{B}\dot{D}} = F_{\dot{D}\dot{B}} \tag{6.43}$$

in the $D(0.1)$ irreducible representation of the same group.

Proof. Since the tensor F_{ij} is skew-symmetric, we have, by (6.40),

$$F_{A\dot{B}C\dot{D}} = -F_{C\dot{D}A\dot{B}}. \tag{6.44}$$

We rewrite this as

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(F_{A\dot{B}C\dot{D}} - F_{C\dot{D}A\dot{B}} = \frac{1}{2}(F_{A\dot{B}C\dot{D}} - F_{C\dot{D}A\dot{B}} + F_{C\dot{D}A\dot{B}} - -F_{C\dot{D}A\dot{B}}). \tag{6.45}$$

Using the fundamental spinor (6.25), we can write (6.45) as follows:

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(\varepsilon_{AC}F_{F\dot{B}}{}^{F}{}_{\dot{D}} + \varepsilon_{\dot{B}\dot{D}}F_{A\dot{E}C}{}^{\dot{E}}). \tag{6.46}$$

Denoting $F_{AC} = (1/2)F_{A\dot{E}C}{}^{\dot{E}}$, we have, by (6.44),

$$F_{AC} = \frac{1}{2}F_{A\dot{E}C}{}^{\dot{E}} = -\frac{1}{2}F_{A}{}^{\dot{E}}{}_{C\dot{E}} = F_{CA}. \tag{6.47}$$

Further, introducing the notation $\overline{F}_{\dot{B}\dot{D}} = \frac{1}{2}F_{F\dot{B}}{}^{F}{}_{\dot{D}}$ and considering that F_{ij} is real, we find

$$\overline{F}_{\dot{B}\dot{D}} = \frac{1}{2}F_{F\dot{B}}{}^{F}{}_{\dot{D}} = \frac{1}{2}\overline{F}_{\dot{F}B}{}^{\dot{F}}{}_{D} = \overline{F}_{BD}. \tag{6.48}$$

Substituting the relationships (6.47) and (6.48) into (6.46), we arrive at (6.44). By definition, the spinor $F_{AC} = F_{CA}$ belongs to the $D(1.0)$ irreducible representation of the groups $SL(2.C)$; and spinor $\overline{F}_{\dot{B}\dot{D}} = \overline{F}_{\dot{D}\dot{B}}$ – to the $D(0.1)$ irreducible representation of the group.

Since the quantities $T_{A\dot{B}C\dot{D}m}$ and $R_{A\dot{B}C\dot{D}kn}$ in the equations (6.37) and (6.38) are skew-symmetric in the pair of spinor matrix indices $A\dot{B}$ and $C\dot{D}$, we can represent them, by (6.27)–(6.28), as

$$T_{A\dot{B}C\dot{D}k} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}T_{ACk} + \varepsilon_{AC}T^{+}_{\dot{B}\dot{D}k}), \tag{6.49}$$

$$R_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}R_{ACkn} + \varepsilon_{AC}R^{+}_{\dot{B}\dot{D}kn}), \tag{6.50}$$

where

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}T_{A\dot{B}C\dot{D}k}, \quad T^{+}_{\dot{B}\dot{D}k} = \frac{1}{2}\varepsilon^{AC}T_{A\dot{B}C\dot{D}k}, \tag{6.51}$$

$$R_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}R_{A\dot{B}C\dot{D}kn}, \quad R^{+}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC}R_{A\dot{B}C\dot{D}kn}. \tag{6.52}$$

In these relationships the + sign with the spinor matrices implies Hermitian conjugation.

6.3 Splitting of structural Cartan equations into irreducible representations of the group SL(2.C)

Matrices (6.51) and (6.52) can be transformed in the spinor indices as follows:

$$T^{A'}_{\ C'k} = S^{A'}_{A} T^{A}_{\ Ck} S^{C}_{C'} + S^{A'}_{A} S^{A}_{\ C',k}, \tag{6.53}$$

$$T^{+\dot{B}'}_{\ \dot{D}'k} = S^{+\dot{B}'}_{\dot{B}} T^{+\dot{B}}_{\ \dot{D}k} S^{+\dot{D}}_{\dot{D}'} + S^{+\dot{B}'}_{\dot{B}} S^{\dot{B}}_{\ \dot{D}',k}, \tag{6.54}$$

$$R^{A'}{}_{C'kn} = S^{A'}_{A} R^{A}{}_{Ckn} S^{C}_{C'}, \tag{6.55}$$

$$R^{+\dot{B}'}{}_{\dot{D}'kn} = S^{+\dot{B}'}_{\dot{B}} R^{+\dot{B}}{}_{\dot{D}kn} S^{+\dot{D}}_{\dot{D}'}. \tag{6.56}$$

Matrices of the transformations $S^{A'}_{A}$ and $S^{+B}_{\dot{B}}$ form the group $SL(2.C)$, and the matrices

$$S^{A'}_{A} \tag{6.57}$$

form the subgroup

$$SL^{+}(2.C) \tag{6.58}$$

of the group $SL(2.C)$, in which the spinors belonging to the irreducible representation $D(r/2,0)$ are transformed.

On the other hand, the matries

$$S^{+\dot{B}'}{}_{\dot{B}} \tag{6.59}$$

form the subgroup

$$SL_{-}(2.C) \tag{6.60}$$

of the group $SL(2.C)$, in which the spinors belonging to the irreducible representation $D(0,s/2)$ are transformed. These properties of the spinors enable the structural Cartan equations to be split into equations that contain spinors transformed in $D(r/2,0)$ or $D(0,s/2)$ irreducible representations of the group $SL(2.C)$.

Proposition 6.3. The second structural Cartan equations (B) in the spinor Δ-basis are split into the equations of the form

$$R_{ACkn} + 2\nabla_{[k} T_{|AC|n]} + 2T_{AE[k} T^{E}_{\ |C|n]} = 0, \tag{6.61}$$

$$R^{+}_{\dot{B}\dot{D}kn} + 2\nabla_{[k} T^{+}_{|\dot{B}\dot{D}|n]} + 2T^{+}_{\dot{B}\dot{F}[k} T^{+\dot{F}}_{\ |\dot{D}|n]} = 0. \tag{6.62}$$

Proof. We write the second structural Cartan equations (6.38) as

$$B_{A\dot{B}C\dot{D}kn} = R_{A\dot{B}C\dot{D}km} + 2\nabla_{[k} T_{|A\dot{B}C\dot{D}|m]} + 2T_{A\dot{B}E\dot{F}[k} T^{E\dot{F}}_{\ |C\dot{D}|m]} = 0. \tag{6.63}$$

Using the fact that the spinor $B_{A\dot{B}C\dot{D}kn}$ is skew-symmetric in the pair of spinor indices $A\dot{B}$ and $\dot{C}D$, we will write it in the form

$$B_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}B_{ACkn} + \varepsilon_{AC}B^{+}_{\dot{B}\dot{D}kn}) = 0, \tag{6.64}$$

where

$$B_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}B_{A\dot{B}C\dot{D}kn} = 0, \tag{6.65}$$

$$B^{+}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC}B_{A\dot{B}C\dot{D}kn} = 0. \tag{6.66}$$

Substituting (6.61) into the equations (6.65) and (6.66) and using the matrices (6.51) and (6.52), we will arrive at the structural equations (6.61) and (6.62) in split form. In the derivation we have used the properties (6.19)-(6.23) of the fundamental spinor ε_{AB}.

Proposition 6.4. Matrices T_{ACk} and $T^{+}_{\dot{B}\dot{D}k}$ in the dyad basis $\xi_{\alpha C}$ have the following form:

$$T_{ACk} = \xi_{\alpha C}\nabla_k \xi^{\alpha}_{A} = T_{ACk}, \tag{6.67}$$

$$T^{+}_{\dot{B}\dot{D}k} = \overline{\xi}_{\dot{\alpha}\dot{D}}\nabla_k \overline{\xi}^{\dot{\alpha}}_{\dot{B}} = T^{+}_{\dot{B}\dot{D}k}. \tag{6.68}$$

Proof. We write the matrices

$$T_{abk} = e^{i}{}_{b}\nabla_k e_{ai}$$

in the spinor basis, using the rules (6.30) and (6.31)

$$T_{A\dot{B}C\dot{D}k} = \sigma^{i}_{C\dot{D}}\nabla_k \sigma_{A\dot{B}i}. \tag{6.69}$$

Substituting this expression into the first one of (6.51) gives

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^{i}_{C\dot{D}}\nabla_k \sigma_{A\dot{B}i}. \tag{6.70}$$

Using the formula (6.6), we write $\sigma_{A\dot{B}i}$ as

$$\sigma_{A\dot{B}i} = \sigma_{\alpha\dot{\beta}i}\xi^{\alpha}_{A}\overline{\xi}^{\dot{\beta}}_{\dot{B}}. \tag{6.71}$$

Substituting (6.71) into (6.70), we have

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^{i}_{C\dot{D}}\nabla_k(\sigma_{\alpha\dot{\beta}i}\xi^{\alpha}_{A}\overline{\xi}^{\dot{\beta}}_{\dot{B}}) = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^{i}_{C\dot{D}}\sigma_{\alpha\dot{\beta}i}\nabla_k(\xi^{\alpha}_{A}\overline{\xi}^{\dot{\beta}}_{\dot{B}}), \tag{6.72}$$

since $\nabla_k(\sigma_{\alpha\dot{\beta}i}) = 0$.

Further, considering that

$$\sigma^{i}_{C\dot{D}}\sigma_{\alpha\dot{\beta}i} = \sigma_{\nu\dot{\gamma}}{}^{i}\xi^{\nu}_{C}\overline{\xi}^{\dot{\gamma}}_{\dot{D}}\sigma_{\alpha\dot{\beta}i} =$$
$$= \delta_{\nu\alpha}\delta_{\dot{\gamma}\dot{\beta}}\xi^{\nu}_{C}\overline{\xi}^{\dot{\gamma}}_{\dot{D}} = \xi_{C\alpha}\overline{\xi}_{\dot{D}\dot{\beta}},$$

we will write (6.72) as

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\xi_{C\alpha}\overline{\xi}_{\dot{D}\dot{\beta}}(\overline{\xi}^{\dot{\beta}}_{\dot{B}}\nabla_k\xi^{\alpha}_{A} + \xi^{\alpha}_{A}\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{B}}). \tag{6.73}$$

In the dyad basis we have the equalities

$$\varepsilon_{\dot{B}\dot{D}} = \overline{\xi}_{\dot{D}\dot{\beta}}\overline{\xi}^{\dot{\beta}}_{\dot{B}}, \quad \varepsilon^{\dot{B}\dot{D}}\overline{\xi}_{\dot{D}\dot{\beta}}\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{B}} = 0,$$

which are conjugates of (6.3) and (6.24). Using these equalities, we can easily obtain (6.67). Similarly, for the conjugate matrix $T^{+}{}_{\dot{B}\dot{D}k}$, we have (6.68).

Proposition 6.5. In the spinor Δ-basis the first structural Cartan equations (A) of the A_4 geometry have the form

$$\nabla_{[k}\sigma^{i]}_{C\dot{D}} - T_{[k|CE}\sigma^{E}{}_{\dot{D}}|^{i]} - \sigma^{[i}_{|C\dot{F}|}T^{+}_{k]\dot{D}}{}^{\dot{F}} = 0 \tag{6.74}$$

or, dropping the matrix indices,

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0. \tag{6.75}$$

Proof. Let us take the derivative $\nabla_k\sigma^{i}_{C\dot{D}}$:

$$\nabla_k\sigma^{i}_{C\dot{D}} = \nabla_k(\sigma^{i}_{\alpha\dot{\beta}}\xi^{\alpha}_{C}\overline{\xi}^{\dot{\beta}}_{\dot{D}}) = \sigma^{i}_{\alpha\dot{\beta}}(\overline{\xi}^{\dot{\beta}}_{\dot{D}}\nabla_k\xi^{\alpha}_{C} + \xi^{\alpha}_{C}\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{D}}).$$

Using (6.67) and (6.68), we will write this relationship as

$$\nabla_k\sigma^{i}_{C\dot{D}} = \sigma^{i}_{\alpha\dot{\beta}}(T_{CEk}\xi^{\alpha E}\overline{\xi}^{\dot{\beta}}_{\dot{D}} + T^{+}_{\dot{D}\dot{F}k}\xi^{\alpha}_{C}\overline{\xi}^{\dot{\beta}\dot{F}}). \tag{6.76}$$

Here we have used the normalization conditions

$$\xi_{\alpha E}\xi^{\alpha E} = 1, \quad \overline{\xi}_{\dot{\beta}\dot{F}}\overline{\xi}^{\dot{\beta}\dot{F}} = 1.$$

Multiplying the terms on the right-hand side (6.76) we obtain, from (6.71),

$$\nabla_k\sigma^{i}_{C\dot{D}} - T_{CEk}\sigma^{iE}_{\dot{D}} - \sigma^{i\dot{F}}_{C}T^{+}_{\dot{D}\dot{F}k} = 0 \tag{6.77}$$

or

$$\nabla_k\sigma^{i}_{C\dot{D}} - T_{kCE}\sigma^{E\,i}_{\dot{D}} - \sigma^{i}_{C\dot{F}}T^{+\dot{F}}_{k\dot{D}}. \tag{6.78}$$

Alternating this relationship in the indices k and i, we obtain the equations (6.74).

Proposition 6.6. The second Bianchi (D) identities of the A_4 geometry in the spinor Δ-basis are split into the following equations:

$$\nabla^{n}\overset{*}{R}_{ACkn} - \overset{*}{R}_{ECkn}\,T^{E}{}_{A}{}^{n} + \overset{*}{R}_{EAkn}\,T^{E}{}_{C}{}^{n} = 0, \tag{6.79}$$

$$\nabla^{n}\overset{*}{R}{}^{+}_{\dot{B}\dot{D}kn} - \overset{*}{R}{}^{+}_{\dot{F}\dot{D}kn}T^{+\dot{F}n}_{\dot{B}} + \overset{*}{R}{}^{+}_{\dot{F}\dot{B}kn}T^{+\dot{F}n}_{\dot{D}} = 0. \tag{6.80}$$

Proof. Increasing and decreasing, using the metric tensors η_{ab} and g_{ik}, the tensor indices in the identities (6.150), we will write them in the form

$$\nabla^n \overset{*}{R}_{abkn} - \overset{*}{R}_{cbkn}\, T^c{}_a{}^n + \overset{*}{R}_{ackn}\, T^c{}_b{}^n = 0. \tag{6.81}$$

In this equality we now pass over to the spinor indices using (6.31) and (6.32) go get

$$\nabla^n \overset{*}{R}_{A\dot{B}C\dot{D}kn} - \overset{*}{R}_{E\dot{F}C\dot{D}kn}\, T^{\dot{E}F}{}_{A\dot{B}}{}^n + \overset{*}{R}_{E\dot{F}A\dot{B}kn}\, T^{\dot{E}F}{}_{C\dot{D}}{}^n = 0. \tag{6.82}$$

We now write this relationship in the form

$$D^n_{A\dot{B}C\dot{D}kn} = 0, \tag{6.83}$$

where by $D^n_{A\dot{B}C\dot{D}kn}$ we have denoted all the terms on the left-hand side of (6.82). Since the relationship (6.83) are skew-symmetrical in the pair of indices $A\dot{B}$ and $C\dot{D}$, we will write it in the form

$$D^n_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}} D^n_{ACkn} + \varepsilon_{AC} D^{+n}{}_{\dot{B}\dot{D}kn}) = 0, \tag{6.84}$$

where

$$D^n_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}} D^n_{A\dot{B}C\dot{D}kn} = 0, \quad D^{+n}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC} D^n_{A\dot{B}C\dot{D}kn} = 0.$$

Substituting here (6.82), we will get (6.79) and (6.80).

Physically, the spinor splitting of the structural Cartan equations (A) and (B) implies splitting into the equations of "matter" and "skew-symmetry", just as it has been done by Dirac in his derivation of equations for the electron and the positron. We can now write equations that are transformed in the groups $SL^+(2.C)$ as

$$\nabla_{[k}\sigma^{i]}_{C\dot{D}} - T_{[k|CE}\sigma^E_{\dot{D}}|^{i]} - \sigma^{[i}{}_{|C\dot{F}|}T^{+}_{k]\dot{D}}{}^{\dot{F}} = 0, \tag{A^s}$$

$$R_{ACkn} + 2\nabla_{[k}T_{|AC|n]} + 2T_{AE[k}T^E_{|C|n]} = 0, \tag{B^s+}$$

and in the group $SL^-(2.C)$ as

$$\nabla_{[k}\sigma^{i]}{}_{C\dot{D}} - T_{[k|CE}\sigma^E_{\dot{D}}|^{i]} - \sigma^{[i}{}_{|C\dot{F}|}T^{+}_{k]\dot{D}}{}^{\dot{F}} = 0, \tag{A^s}$$

$$R^{+}_{\dot{B}\dot{D}kn} + 2\nabla_{[k}T^{+}_{|\dot{B}\dot{D}|n]} + 2T^{+}_{\dot{B}\dot{F}[k}T^{+\dot{F}}_{|\dot{D}|n]} = 0. \tag{B^s-}$$

In the numerations of these formulas s implies transformation in a spinor group. Dropping the matrix indices, we will write these relationships as

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0, \tag{A^s}$$

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0, \tag{B^s+}$$

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0, \qquad (A^s)$$

$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_{k}, T^{+}_{n}] = 0. \qquad (B^{s-})$$

Correspondingly, discarding the matrix indices in the equations (6.79) and (6.80), we obtain

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \qquad (D^{s+})$$

$$\nabla^n \overset{*}{R}{}^{+}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \qquad (D^{s-})$$

6.4 Carmeli matrices

Equalities (6.67) and (6.68) can be written in matrix form

$$T_k = \xi \nabla_k \xi, \tag{6.85}$$

$$T^{+}_{k} = \xi^{+} \nabla_k \xi^{+}, \tag{6.86}$$

where T_k and ξ are 2×2 complex matrices with elements $T^A{}_{Bk}$ and ξ^a_A, respectively. Multiplying T_k by $\sigma^k{}_{A\dot{B}}$, we can introduce the traceless Carmeli 2×2 matrices [44-46]

$$T_{A\dot{B}} = \sigma^k_{A\dot{B}} T_k, \tag{6.87}$$

$$A, C \ldots = 0, 1, \quad \dot{B}, \dot{D} \ldots = \dot{0}, \dot{1}$$

with the components

$$T_{0\dot{0}} = \begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}, \quad T_{0\dot{1}} = \begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix},$$
$$T_{1\dot{0}} = \begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix}, \quad T_{1\dot{1}} = \begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix}. \tag{6.88}$$

Using matrices (6.87), we can define the matrix elements

$(T_{A\dot{B}})_C{}^D =$ (6.89)

$A\dot{B}$ \ CD	00	01	10	11
$0\dot{0}$	ε	$-\kappa$	π	$-\varepsilon$
$0\dot{1}$	β	$-\sigma$	μ	$-\beta$
$1\dot{0}$	α	$-\rho$	λ	$-\alpha$
$1\dot{1}$	γ	$-\tau$	ν	$-\gamma$

where $(T_{A\dot{B}})_C{}^D$ is the CD element of the matrices $T_{A\dot{B}}$. Consequently, the complex conjugate matrices $T^{+}{}_{\dot{A}B}$ are

$(T^{+}_{\dot{A}B})^{\dot{D}}{}_{\dot{C}} =$ (6.90)

$\dot{A}B$ \ $\dot{C}\dot{D}$	$\dot{0}\dot{0}$	$\dot{0}\dot{1}$	$\dot{1}\dot{0}$	$\dot{1}\dot{1}$
$\dot{0}0$	$\overline{\varepsilon}$	$-\overline{\kappa}$	$\overline{\pi}$	$-\overline{\varepsilon}$
$\dot{0}1$	$\overline{\beta}$	$-\overline{\sigma}$	$\overline{\mu}$	$-\overline{\beta}$
$\dot{1}0$	$\overline{\alpha}$	$-\overline{\rho}$	$\overline{\lambda}$	$-\overline{\alpha}$
$\dot{1}1$	$\overline{\gamma}$	$-\overline{\tau}$	$\overline{\nu}$	$-\overline{\gamma}$

Proposition 6.7. In the Carmeli matrices the first structural Cartan equations (A) of the A_4 geometry have the form

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - \\ -(T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \tag{6.91}$$

Proof. We will write the equations (6.75) as

$$\nabla_k\sigma^i_{C\dot{D}} - \nabla_k\sigma^i_{A\dot{B}} = T_C{}^E{}_k\sigma^i_{\dot{D}E} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k - \\ -T_A{}^C{}_k\sigma^i_{C\dot{B}} - \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}\,k}. \tag{6.92}$$

It is easily seen that the equations (6.92) represent the difference of the two relationships

$$\nabla_k\sigma^i_{C\dot{D}} = T_C{}^E{}_k\sigma^i_{\dot{D}E} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k, \tag{6.93}$$

$$\nabla_k\sigma^i_{A\dot{B}} = T_A{}^C{}_k\sigma^i_{C\dot{B}} + \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}}{}_k. \tag{6.94}$$

Multiplying (6.93) by $\sigma^k{}_{A\dot{B}}$, and (6.94) by $\sigma^k{}_{C\dot{D}}$, we get

$$\nabla_k\sigma^i_{C\dot{D}}\sigma^k_{A\dot{B}} = T_C{}^E{}_k\sigma^i_{\dot{D}E}\sigma^k_{A\dot{B}} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k\sigma^k_{A\dot{B}}, \tag{6.95}$$

$$\nabla_k\sigma^i_{A\dot{B}}\sigma^k_{C\dot{D}} = T_A{}^P{}_k\sigma^i_{P\dot{B}}\sigma^k_{C\dot{D}} + \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}}{}_k\sigma^k_{C\dot{D}}. \tag{6.96}$$

We now introduce the notation

$$(T_{A\dot{B}})_C{}^E = T_C{}^E{}_k\sigma^k_{A\dot{B}} \tag{6.97}$$

and

$$\partial_{A\dot{B}} = \sigma^k_{A\dot{B}}\nabla_k, \tag{6.98}$$

and rewrite the relationships (6.95) and (6.96) as

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}}, \tag{6.99}$$

$$\partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} + \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \tag{6.100}$$

Subtracting from (6.99) the equality (6.100), we will arrive at the first structural Cartan equations (6.91) of the A_4 geometry, written in terms of Carmeli matrices.

Consider now the second structural Cartan equations (B^{s+}), written in matrix forms

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0. \tag{6.101}$$

Multiplying the quantity R_{kn} by $\sigma^k{}_{A\dot{B}}$ and $\sigma^n{}_{C\dot{D}}$, we will introduce the traceless Carmeli matrix

$$R_{A\dot{B}C\dot{D}} = R_{kn}\sigma^k_{A\dot{B}}\sigma^n_{C\dot{D}} \tag{6.102}$$

with the components [44-46]

$$R_{0\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 + 2\Lambda & -\Psi_1 \end{pmatrix}, \quad R_{1\dot{0}0\dot{0}} = \begin{pmatrix} \Phi_{10} & -\Phi_{00} \\ \Phi_{20} & -\Phi_{10} \end{pmatrix},$$
$$R_{1\dot{1}1\dot{0}} = \begin{pmatrix} \Psi_3 & -\Psi_2 - 2\Lambda \\ \Psi_4 & -\Psi_3 \end{pmatrix}, \quad R_{1\dot{1}0\dot{1}} = \begin{pmatrix} \Phi_{12} & -\Phi_{02} \\ \Phi_{22} & -\Phi_{12} \end{pmatrix},$$
$$R_{1\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_2 + \Phi_{11} - \Lambda & -\Psi_1 - \Phi_{01} \\ \Psi_3 + \Phi_{21} & -\Psi_2 - \Phi_{11} + \Lambda \end{pmatrix},$$
$$R_{1\dot{0}0\dot{1}} = \begin{pmatrix} -\Psi_2 + \Phi_{11} + \Lambda & \Psi_1 - \Phi_{01} \\ -\Psi_3 + \Phi_{21} & \Psi_2 - \Phi_{11} - \Lambda \end{pmatrix}. \tag{6.103}$$

Proposition 6.8. In terms of Carmeli spinor matrices (6.87) and (6.102), the second structural Cartan equations (B^{s+}) of the A_4 geometry become

$$R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}} T_{A\dot{B}} - \partial_{A\dot{B}} T_{C\dot{D}} - (T_{C\dot{D}})_A{}^F T_{F\dot{B}} - (T^+_{\dot{D}C})^{\dot{F}}{}_{\dot{B}} T_{A\dot{F}} + $$
$$+ (T_{A\dot{B}})_C{}^F T_{F\dot{D}} + (T^+_{\dot{B}A})^{\dot{F}}{}_{\dot{D}} T_{C\dot{F}} + [T_{A\dot{B}}, T_{C\dot{D}}]. \tag{6.104}$$

Proof. We write the equations (6.101) as

$$R_{kn} = 2\nabla_{[n} T_{k]} + [T_k, T_n] \tag{6.105}$$

or

$$R_{kn} = \nabla_n T_k - \nabla_k T_n + T_k T_n - T_n T_k. \tag{6.106}$$

Multiplying this by $\sigma^k{}_{A\dot{B}} \sigma^n{}_{C\dot{D}}$, we will have

$$R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}} T_k \sigma^k_{A\dot{B}} - \partial_{A\dot{B}} T_n \sigma^n_{C\dot{D}} + T_{A\dot{B}} T_{C\dot{D}} - T_{C\dot{D}} T_{A\dot{B}} =$$
$$= \partial_{C\dot{D}} T_{A\dot{B}} - \partial_{AB} T_{C\dot{D}} - (\partial_{C\dot{D}} \sigma_{A\dot{B}}{}^k - \partial_{A\dot{B}} \sigma_{C\dot{D}}{}^k) T_k +$$
$$+ [T_{A\dot{B}}, T_{C\dot{D}}]. \tag{6.107}$$

We have used here the condition that

$$\sigma_k{}^{A\dot{B}} \sigma^k_{C\dot{E}} = \delta^A_C \delta^{\dot{B}}_{\dot{E}} \tag{6.108}$$

and the notation

$$\partial_{A\dot{B}} = \sigma^k_{A\dot{B}} \nabla_k. \tag{6.109}$$

If now in (6.107) we use the relationships (6.99) and (6.100), we will get the equations (6.103).

Let us write the second Bianchi identities (D^{s+}) of the A_4 geometry in matrix form

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0. \tag{6.110}$$

Multiplying these equations by $\sigma^n{}_{E\dot{F}}$, we will render them in terms of Carmeli matrices as follows:

$$\partial^{C\dot{D}} \overset{*}{R}_{E\dot{F}C\dot{D}} + \sigma^n_{\dot{E}F} (\nabla^{C\dot{D}} \sigma^{A\dot{B}}_n) \overset{*}{R}_{A\dot{B}C\dot{D}} +$$
$$+ (\nabla_k \sigma^{kC\dot{D}}) \overset{*}{R}_{E\dot{F}C\dot{D}} - [T^{C\dot{D}}, \overset{*}{R}_{E\dot{F}C\dot{D}}] = 0. \tag{6.111}$$

Using the relationship (6.99), we can rewrite the identities (6.111) as

$$\begin{aligned} \partial^{C\dot{D}} \overset{*}{R}_{E\dot{F}C\dot{D}} - (T^{C\dot{D}})^{A}{}_{E} \overset{*}{R}_{A\dot{F}C\dot{D}} - \\ -(T^{+\dot{D}C})_{\dot{F}}{}^{\dot{B}} \overset{*}{R}_{E\dot{B}C\dot{D}} + (T_{P}{}^{\dot{D}})^{CP} \overset{*}{R}_{E\dot{F}C\dot{D}} + \\ +(T^{+C}_{\dot{Q}})^{\dot{Q}\dot{D}} \overset{*}{R}_{E\dot{F}C\dot{D}} + [T^{C\dot{D}}, \overset{*}{R}_{E\dot{F}C\dot{D}}] = 0. \end{aligned} \tag{6.112}$$

6.5 Component-by-component rendering of structural equations of A$_4$ geometry

Let us now write the equations (6.91) component by component. For convenience, we will introduce the following notation:

$$\begin{aligned} A^{i}{}_{C\dot{D}A\dot{B}} = \partial_{C\dot{D}}\sigma^{i}{}_{A\dot{B}} - \partial_{A\dot{B}}\sigma^{i}{}_{C\dot{D}} = (T_{C\dot{D}})_{A}{}^{P}\sigma^{i}{}_{P\dot{B}} + \\ +\sigma^{i}{}_{A\dot{R}}(T^{+}_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - (T_{A\dot{B}})_{C}{}^{P}\sigma^{i}{}_{P\dot{D}} - \sigma^{i}{}_{C\dot{R}}(T^{+}_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \end{aligned} \tag{6.113}$$

Also, we will denote the components of the spinor derivative as

$$\partial_{A\dot{B}} = \begin{array}{c|cc} & \multicolumn{2}{c}{\dot{B}} \\ A & \dot{0} & \dot{1} \\ \hline 0 & D & \delta \\ 1 & \bar{\delta} & \Delta \end{array}, \tag{6.114}$$

and the components of the spinor Δ-basis as

$$\sigma^{i}{}_{A\dot{B}} = \begin{array}{c|cc} & \multicolumn{2}{c}{\dot{B}} \\ A & \dot{0} & \dot{1} \\ \hline 0 & l^{i} = (Y^{0}, V, Y^{2}, Y^{3}) & m^{i} = (\xi^{0}, \omega, \xi^{2}, \xi^{3}) \\ 1 & \overline{m}^{i} = (\overline{\xi}^{0}, \overline{\omega}, \overline{\xi}^{2}, \overline{\xi}^{3}) & n^{i} = (X^{\circ}, U, X^{2}, X^{3}) \end{array}. \tag{6.115}$$

From (6.113), the spinor component $A^{i}{}_{0\dot{0}0\dot{1}}$ will be

$$\begin{aligned} A^{i}{}_{0\dot{0}0\dot{1}} = \partial_{0\dot{0}}\sigma^{i}{}_{0\dot{1}} - \partial_{0\dot{1}}\sigma^{i}{}_{0\dot{0}} = (T_{0\dot{0}})_{0}{}^{P}\sigma^{i}{}_{P\dot{1}} + \sigma^{i}{}_{0\dot{R}}(T^{+}_{\dot{0}0})^{\dot{R}}{}_{\dot{1}} - \\ -(T_{0\dot{1}})_{0}{}^{P}\sigma^{i}{}_{P\dot{0}} - \sigma^{i}{}_{0\dot{R}}(T^{+}_{\dot{1}0})^{\dot{R}}{}_{\dot{0}} \end{aligned} \tag{6.116}$$

or

$$\begin{aligned} A^{i}{}_{0\dot{0}0\dot{1}} = \partial_{0\dot{0}}\sigma^{i}{}_{0\dot{1}} - \partial_{0\dot{1}}\sigma^{i}{}_{0\dot{0}} = \left((T_{0\dot{0}})_{0}{}^{0}\sigma^{i}{}_{0\dot{1}} + (T_{0\dot{0}})_{0}{}^{1}\sigma^{i}{}_{1\dot{1}}\right) + \\ +\left(\sigma^{i}{}_{0\dot{0}}(T^{+}_{\dot{0}0})^{\dot{0}}{}_{\dot{1}} + \sigma^{i}{}_{0\dot{1}}(T^{+}_{\dot{0}0})^{\dot{1}}{}_{\dot{1}}\right) - \left((T_{0\dot{1}})_{0}{}^{0}\sigma^{i}{}_{0\dot{0}} + (T_{0\dot{1}})_{0}{}^{1}\sigma^{i}{}_{1\dot{0}}\right) - \\ -\left(\sigma^{i}{}_{0\dot{0}}(T^{+}_{\dot{1}0})^{\dot{0}}{}_{\dot{0}} + \sigma^{i}{}_{0\dot{1}}(T^{+}_{\dot{1}0})^{\dot{1}}{}_{\dot{0}}\right). \end{aligned} \tag{6.117}$$

Using the notation of (6.89)-(6.90) and (6.114)-(6.115) for the components $(T_{C\dot{D}})_A{}^P, (T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}, \partial_{A\dot{B}}$ and $\sigma^i{}_{A\dot{B}}$, we will obtain, by (6.117),

$$\begin{aligned} Dm^i - \delta l^i = \left(\varepsilon m^i + (-\kappa) n^i\right) + \left(l^i \overline{\pi} + m^i(-\overline{\varepsilon})\right) - \\ - \left(\beta l^i + (-\sigma)\overline{m}^i\right) - \left(l^i \overline{\alpha} + m^i(-\overline{\rho})\right) = \\ = -(\overline{\alpha} + \beta - \overline{\pi}) l^i - \kappa n^i + \sigma \overline{m}^i + (\overline{\rho} + \varepsilon - \overline{\varepsilon}) m^i. \end{aligned} \tag{6.118}$$

Since the vectors m^i and l^i have the following components:

$$l^i = (Y^0, V, Y^2, Y^3), \quad m^i = (\xi^0, \omega, \xi^2, \xi^3),$$

it follows from (6.118) that

$$\delta V - D\omega = (\overline{\alpha} + \beta - \overline{\pi}) V + \kappa U - \sigma \overline{\omega} - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\omega, \tag{6.119}$$

$$\delta Y^\alpha - D\xi^\alpha = (\overline{\alpha} + \beta - \overline{\pi}) Y^\alpha + \kappa X^\alpha - \alpha \overline{\xi}^\alpha - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\xi^\alpha, \tag{6.120}$$

$$\alpha = 0, 2, 3.$$

In a similar manner we find the following component rendering of the first structural Cartan equations of the A_4 geometry

$$\delta V - D\omega = (\overline{\alpha} + \beta - \overline{\pi}) V + \kappa U - \sigma \overline{\omega} - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\omega, \tag{A.1}$$

$$\delta Y^\alpha - D\xi^\alpha = (\overline{\alpha} + \beta - \overline{\pi}) Y^\alpha + \kappa X^\alpha - \sigma \overline{\xi}^\alpha - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\xi^\alpha, \tag{A.2}$$

$$\Delta Y^\alpha - D X^\alpha = (\gamma + \overline{\gamma}) Y^\alpha + (\varepsilon + \overline{\varepsilon}) X^\alpha - (\tau + \overline{\pi})\xi^\alpha - (\overline{\tau} + \pi)\omega, \tag{A.3}$$

$$\Delta V - DV = (\gamma + \overline{\gamma}) V + (\varepsilon + \overline{\varepsilon}) U - (\tau + \overline{\pi})\overline{\omega} - (\overline{\tau} + \pi)\omega, \tag{A.4}$$

$$\delta U - \Delta\omega = -\overline{\nu} V + (\tau - \overline{\alpha} - \beta) U + \overline{\lambda \omega} + (\mu - \gamma + \overline{\gamma})\omega, \tag{A.5}$$

$$\delta X^\alpha - \Delta\xi^\alpha = -\overline{\nu} Y^\alpha + (\tau - \overline{\alpha} - \beta) X^\alpha + \overline{\lambda \xi}^\alpha + (\mu - \gamma + \overline{\gamma})\xi^\alpha, \tag{A.6}$$

$$\overline{\delta}\omega - \delta\overline{\omega} = (\overline{\mu} - \mu) V + (\overline{\rho} - \rho) U - (\overline{\alpha} - \beta)\overline{\omega} - (\overline{\beta} - \alpha)\omega, \tag{A.7}$$

$$\overline{\delta}\xi^\alpha - \delta\overline{\xi}^\alpha = (\overline{\mu} - \mu) Y^\alpha + (\overline{\rho} - \rho) X^\alpha - (\overline{\alpha} - \beta)\overline{\xi}^\alpha - (\overline{\beta} - \alpha)\xi^\alpha, \tag{A.8}$$

$$\alpha = 0, 2, 3,$$

and the complex conjugate equations $(\overline{A}.1) - (\overline{A}.8)$ (all in all 24 independent equations).

Let us now look at the equations (6.107) and write them componentwise. For instance, we will derive the $R_{0\mathtt{i}0\dot{0}}$ component of these equations

$$\begin{aligned} R_{0\mathtt{i}0\dot{0}} \;=\; & \partial_{0\dot{0}} T_{0\mathtt{i}} - \partial_{0\mathtt{i}} T_{0\dot{0}} - (T_{0\dot{0}})_0{}^0 T_{0\mathtt{i}} - (T_{0\dot{0}})_0{}^1 T_{1\mathtt{i}} + \\ & + (T^+_{0\dot{0}})^{\dot{0}}{}_{\mathtt{i}} T_{0\dot{0}} - (T^+_{0\dot{0}})^{\mathtt{i}}{}_{\mathtt{i}} T_{0\mathtt{i}} + (T_{0\mathtt{i}})_0{}^0 T_{0\dot{0}} + (T_{0\mathtt{i}})_0{}^1 T_{1\dot{0}} + \\ & + (T^+_{1\dot{0}})^{\dot{0}}{}_{\dot{0}} T_{0\dot{0}} + (T^+_{1\dot{0}})^{\mathtt{i}}{}_{\dot{0}} T_{0\mathtt{i}} + T_{0\mathtt{i}} T_{0\dot{0}} - T_{0\dot{0}} T_{1\dot{0}}. \end{aligned} \tag{6.121}$$

Using the matrices (6.89)-(6.90), (6.103) and the spinor derivative (6.114), we can represent (6.121) as

$$
\begin{aligned}
\begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 + 2\Lambda & -\Psi_1 \end{pmatrix} &= D\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} - \delta\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} - \\
-\varepsilon\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} &+ \kappa\begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix} - \overline{\pi}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} + \\
+\overline{\varepsilon}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} &+ \beta\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} - \sigma\begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix} + \\
+\overline{\alpha}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} &- \overline{\rho}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} + \\
+\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} &- \begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}.
\end{aligned}
\tag{6.122}
$$

These equations split into the following three independent equations:

$$
\begin{aligned}
(D - \overline{\rho} + \overline{\varepsilon})\beta - (\delta - \overline{\alpha} + \overline{\pi})\varepsilon - (\alpha + \pi)\sigma + (\mu + \gamma)\kappa - \Psi_1 = 0, \\
(D - \rho - \overline{\rho} - 3\varepsilon + \overline{\varepsilon})\sigma - (\delta - \tau + \overline{\pi} - \overline{\alpha} - 3\beta)\kappa - \Psi_0 = 0, \\
(D - \overline{\rho} + \varepsilon + \overline{\varepsilon})\mu - (\delta + \overline{\pi} - \overline{\alpha} + \beta)\pi - \sigma\lambda + \nu\kappa - 2\Lambda - \Psi_2 = 0.
\end{aligned}
$$

Similarly, we will obtain the following independent equations (B^{s+}):

$$(D - \rho - \varepsilon - \overline{\varepsilon})\rho - (\overline{\delta} - 3\alpha - \overline{\beta} + \pi)\kappa -$$
$$-\sigma\overline{\sigma} + \tau\overline{\kappa} - \Phi_{00} = 0, \tag{$B^{s+}.1$}$$

$$(D - \rho - \overline{\rho} - 3\varepsilon + \overline{\varepsilon})\sigma - (\delta - \tau + \overline{\pi} - \overline{\alpha} - 3\beta)\kappa -$$
$$-\Psi_0 = 0, \tag{$B^{s+}.2$}$$

$$(D - \rho - \varepsilon + \overline{\varepsilon})\tau - (\Delta - 3\gamma - \overline{\gamma})\kappa - \rho\overline{\pi} - \sigma\overline{\tau} - \pi\sigma -$$
$$-\Psi_1 - \Phi_{10} = 0, \tag{$B^{s+}.3$}$$

$$(D - \rho - \overline{\varepsilon} + 2\varepsilon)\alpha - (\overline{\delta} - \overline{\beta} + \pi)\varepsilon - \beta\overline{\sigma} + \kappa\lambda + \overline{\kappa}\gamma -$$
$$-\pi\rho - \Phi_{10} = 0, \tag{$B^{s+}.4$}$$

$$(D + \varepsilon + \overline{\varepsilon})\gamma - (\Delta - \gamma - \overline{\gamma})\varepsilon - (\tau + \overline{\pi})\alpha - (\pi + \overline{\tau})\beta -$$
$$-\pi\tau + \nu\kappa + \Lambda - \Psi_2 - \Phi_{11} = 0, \tag{$B^{s+}.5$}$$

$$(D - \rho + 3\varepsilon - \overline{\varepsilon})\lambda - (\overline{\delta} + \pi + \alpha - \overline{\beta})\pi - \mu\overline{\sigma} + \nu\overline{\kappa} - \Phi_{20} = 0, \tag{$B^{s+}.6$}$$

$$(D - \overline{\rho} + \overline{\varepsilon})\beta - (\delta - \overline{\alpha} + \overline{\pi})\varepsilon - (\alpha + \pi)\sigma + (\mu + \gamma)\kappa -$$
$$-\Psi_1 = 0, \tag{$B^{s+}.7$}$$

$$(D - \overline{\rho} + \varepsilon + \overline{\varepsilon})\mu - (\delta + \overline{\pi} - \overline{\alpha} + \beta)\pi - \sigma\lambda + \nu\kappa -$$
$$-2\Lambda - \Psi_2 = 0, \tag{$B^{s+}.8$}$$

$$(D + 3\varepsilon + \overline{\varepsilon})\nu - (\Delta + \mu + \gamma - \overline{\gamma})\pi - \mu\overline{\tau} - (\overline{\pi} + \tau)\lambda -$$

$$-\Psi_3 - \Phi_{21} = 0, \qquad (B^{s+}.9)$$

$$(\Delta + \mu + \overline{\mu} + 3\gamma - \overline{\gamma})\lambda - (\overline{\delta} + 3\alpha + \overline{\beta} + \pi - \overline{\tau})\nu + \Psi_4 = 0, \qquad (B^{s+}.10)$$

$$(\delta - \overline{\alpha} - \beta - \tau)\rho - (\delta - 3\alpha + \overline{\beta})\sigma + \tau\overline{\rho} - (\mu - \overline{\mu})\kappa +$$
$$+\Psi_1 - \Phi_{01} = 0, \qquad (B^{s+}.11)$$

$$(\delta - \overline{\alpha} + 2\beta)\alpha - (\overline{\delta} + \overline{\beta})\beta - \mu\rho + \sigma\lambda - (\rho - \overline{\rho})\gamma -$$
$$-(\mu - \overline{\mu})\varepsilon - \Lambda + \Psi_2 - \Phi_{11} = 0, \qquad (B^{s+}.12)$$

$$(\delta - \overline{\alpha} + 3\beta)\lambda - (\overline{\delta} + \pi + \alpha + \overline{\beta})\mu - (\rho - \overline{\rho})\nu +$$
$$+\pi\overline{\mu} + \Psi_3 - \Phi_{21} = 0, \qquad (B^{s+}.13)$$

$$(\delta - \tau + \overline{\alpha} + \beta)\gamma - (\Delta - \gamma + \overline{\gamma} + \mu)\beta - \mu\tau + \sigma\nu +$$
$$+\varepsilon\overline{\nu} - \alpha\overline{\lambda} - \Phi_{12} = 0, \qquad (B^{s+}.14)$$

$$(\delta - \tau + 3\beta + \overline{\alpha})\nu - (\Delta + \mu + \gamma + \overline{\gamma})\mu - \lambda\overline{\lambda} +$$
$$+\pi\overline{\nu} - \Phi_{22} = 0, \qquad (B^{s+}.15)$$

$$(\delta - \tau - \beta + \overline{\alpha})\tau - (\Delta + \mu - 3\gamma + \overline{\gamma})\sigma - \overline{\lambda}\rho +$$
$$+\kappa\overline{\nu} - \Phi_{02} = 0, \qquad (B^{s+}.16)$$

$$(\Delta + \overline{\mu} - \gamma - \overline{\gamma})\rho - (\overline{\delta} + \overline{\beta} - \alpha - \overline{\tau})\tau + \sigma\lambda -$$
$$-\nu\kappa + 2\Lambda + \Psi_2 = 0, \qquad (B^{s+}.17)$$

$$(\Delta - \overline{\gamma} + \overline{\mu})\alpha - (\overline{\delta} + \overline{\beta} - \overline{\tau})\gamma - (\rho + \varepsilon)\nu +$$
$$+(\tau + \beta)\lambda + \Psi_3 = 0. \qquad (B^{s+}.18)$$

In addition to these equations, the second structural Cartan equations (B) include the complex conjugate equations

$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_{k}, T^{+}_{n}] = 0. \qquad (B^{s-})$$

We can write these equations in terms of components by replacing the equations $(B^{s+}.1)$–$(B^{s+}.18)$ by their complex conjugate equations.

6.6 Connection of structural Cartan equations of A_4 geometry with the NP formalism

In 1962 Newman and Penrose [40] put forward a system of nonlinear spinor equations, which appeared to be extremely convenient in the search for novel solutions of Einstein's equations. In the work [47] by the author of this book is was shown that the equations of the Newman-Penrose formalism coincide with the structural Cartan equations of the geometry of absolute parallelism. Indeed, with spinor Carmeli matrices $T_{C\dot{D}}$ one can connect the spintensor $T_{FAC\dot{D}}$ using the relationships

$$(T_{C\dot{D}})_A{}^P = T_A{}^P{}_k\sigma^k{}_{C\dot{D}} = T^P{}_{AC\dot{D}} = -\varepsilon^{PF}T_{FAC\dot{D}}. \qquad (6.123)$$

Using the matrix elements (6.162) of the Carmeli matrices and the fundamental spinor

$$\varepsilon^{AB} = \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we will obtain the following notation for the components of the spintensor $T_{ABC\dot{D}}$:

$$T_{ABC\dot{D}} = \begin{array}{c|cccc} & \multicolumn{4}{c}{C\dot{D}} \\ AB & 0\dot{0} & 0\dot{1} & 1\dot{0} & 1\dot{1} \\ \hline 00 & \kappa & \sigma & \rho & \tau \\ (01) & \varepsilon & \beta & \alpha & \gamma \\ 11 & \pi & \mu & \lambda & \nu \end{array}. \qquad (6.124)$$

Proposition 6.9. First structural Cartan equations of the A_4 geometry coincide with the "coordinate equations" [40]

$$\partial_{A\dot{B}}\sigma^{i}_{\ C\dot{D}} - \partial_{C\dot{D}}\sigma^{i}_{\ A\dot{B}} = \varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^{i}_{\ \dot{Q}B} - T_{PCA\dot{B}}\sigma^{i}_{\ \dot{Q}D}) + \\ +\varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^{i}_{\ A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^{i}_{\ C\dot{S}}) \qquad (6.125)$$

in the Newman-Penrose formalism.

Proof. We will write the structural Cartan equations (A) of the geometry of absolute parallelism as

$$\partial_{C\dot{D}}\sigma^{i}_{\ A\dot{B}} - \partial_{A\dot{B}}\sigma^{i}_{\ C\dot{D}} = (T_{C\dot{D}})_{A}{}^{P}\sigma^{i}_{\ P\dot{B}} + \sigma^{i}_{\ A\dot{R}}(T^{+}_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - \\ -(T_{A\dot{B}})_{C}{}^{P}\sigma^{i}_{\ P\dot{D}} - \sigma^{i}_{\ C\dot{R}}(T^{+}_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \qquad (6.126)$$

Using the relationship (6.123), we will represent the equations (6.126) as

$$\partial_{C\dot{D}}\sigma^{i}_{\ A\dot{B}} - \partial_{A\dot{B}}\sigma^{i}_{\ C\dot{D}} = -\big(\varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^{i}_{\ \dot{Q}B} - T_{PCA\dot{B}}\sigma^{i}_{\ \dot{Q}D}) + \\ +\varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^{i}_{\ A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^{i}_{\ C\dot{S}})\big).$$

It is easily seen that these equations are equivalent to (6.125).

We now write the well-known decomposition of the Riemannian tensor R_{ijkm} into irreducible representations

$$R_{ijkm} = C_{ijkm} - 2g_{[i[k}R_{m]j]} - \frac{1}{3}Rg_{i[m}g_{k]j}, \qquad (6.127)$$

where C_{ijkm} is the Weyl tensor (10 independent coordinates); R_{ij} is the Ricci tensor (nine independent coordinates); R is the scalar curvature. The spinor representation of these quantities using the Newman-Penrose formalism looks like [48]

$$C_{ijkm} \leftrightarrow \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \qquad (6.128)$$

$$R_{ij} \leftrightarrow 2\Phi_{AB\dot{A}\dot{B}} + 6\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}, \qquad (6.129)$$

$$R = 24\Lambda, \qquad (6.130)$$

where spinors Ψ_{ABCD} and $\Phi_{AB\dot{A}\dot{B}}$ have the following symmetry properties:

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)\dot{A}\dot{B}}. \tag{6.131}$$

By definition the spinors Ψ_{ABCD} and $\Phi_{AB\dot{A}\dot{B}}$ are transformed following the $D(2.0)$ and $D(1.1)$ irreducible representation of the groups $SL^+(2.C)$, respectively.

If we now put in juxtaposition to the Riemann tensor R_{ijkm} a spintensor following the rule

$$R_{ijkm} \leftrightarrow R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}},$$

then in terms of the spinors (6.128)-(6.130) it can be written as

$$\begin{aligned} R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = {} & \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \\ & +\Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}} + \overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + \\ & +2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{C}\dot{D}} + \\ & +\varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}). \end{aligned} \tag{6.132}$$

This spintensor being skew-symmetric in the pair of indices $A\dot{A}$ and $B\dot{B}$, we will write it as

$$R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}} = \frac{1}{2}(\varepsilon_{\dot{E}\dot{B}}R_{ACD\dot{P}F\dot{Q}} + \varepsilon_{AC}\overline{R}_{\dot{E}\dot{B}D\dot{P}F\dot{Q}}), \tag{6.133}$$

where

$$R_{ACD\dot{P}F\dot{Q}} = \frac{1}{2}\varepsilon^{\dot{E}\dot{B}}R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}}, \tag{6.134}$$

$$\overline{R}_{\dot{E}\dot{B}D\dot{P}F\dot{Q}} = \frac{1}{2}\varepsilon^{AC}R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}}. \tag{6.135}$$

Substituting into these relationships the equality (6.132) gives

$$R_{ACD\dot{P}F\dot{Q}} = \Psi_{ACDF}\varepsilon_{\dot{P}\dot{Q}} + \Phi_{AC\dot{Q}\dot{P}}\varepsilon_{FD} + \Lambda\varepsilon_{\dot{P}\dot{Q}}(\varepsilon_{CD}\varepsilon_{AF} + \varepsilon_{AD}\varepsilon_{CF}), \tag{6.136}$$

$$\overline{R}_{\dot{E}\dot{B}D\dot{B}PQ} = \varepsilon_{DP}\overline{\Psi}_{\dot{E}\dot{B}\dot{P}\dot{Q}} + \overline{\Phi}_{\dot{B}\dot{E}PD}\varepsilon_{\dot{Q}\dot{P}} + \Lambda\varepsilon_{DP}(\varepsilon_{\dot{B}\dot{P}}\varepsilon_{\dot{E}\dot{Q}} + \varepsilon_{\dot{E}\dot{P}}\varepsilon_{\dot{B}\dot{Q}}). \tag{6.137}$$

Proposition 6.10. The second structural Cartan equations (B^{s+}) are equivalent to the equations [40]

$$\begin{aligned} & \Psi_{ACDF}\varepsilon_{\dot{E}\dot{B}} + \Phi_{AC\dot{B}\dot{E}}\varepsilon_{FD} + \Lambda\varepsilon_{\dot{E}\dot{B}}(\varepsilon_{CD}\varepsilon_{AF} + \\ & +\varepsilon_{AD}\varepsilon_{CF}) - \partial_{D\dot{B}}T_{ACF\dot{E}} + \partial_{F\dot{E}}T_{ACD\dot{B}} + \\ & +\varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} + T_{ACP\dot{B}}T_{QDF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}} - \\ & -T_{ACP\dot{E}}T_{QFD\dot{B}}) + \\ & +\varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}}\overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}}\overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0 \end{aligned} \tag{6.138}$$

in the Newman-Penrose formalism.
Proof. We write the equations (B^{s+}) in terms of the Carmeli matrices

$$\begin{aligned} R_{F\dot{E}D\dot{B}} = \partial_{D\dot{B}}T_{F\dot{E}} - \partial_{F\dot{E}}T_{D\dot{B}} - (T_{D\dot{B}})_F{}^S T_{S\dot{B}} - \\ -(T^+_{\dot{E}D})^{\dot{F}}{}_{\dot{B}}T_{F\dot{F}} + (T_{F\dot{E}})_D{}^S T_{S\dot{B}} + \\ +(T^+_{\dot{E}F})^{\dot{F}}{}_{\dot{B}}T_{D\dot{F}} + [T_{F\dot{E}}, T_{D\dot{B}}]. \end{aligned} \tag{6.139}$$

Using the relationships (6.123), we can represent the equations (6.139) as

$$\begin{aligned} R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}}T_{ACE\dot{F}} + \partial_{E\dot{F}}T_{ACD\dot{B}} + T^S{}_{FD\dot{B}}T_{ACS\dot{E}} + \\ +\overline{T}^{\dot{F}}{}_{\dot{E}\dot{B}D}T_{ACF\dot{F}} - T^S{}_{DF\dot{E}}T_{ACS\dot{B}} - \overline{T}^{\dot{F}}{}_{\dot{B}\dot{E}F}T_{ACD\dot{F}} + \\ +\varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}}) = 0, \end{aligned}$$

or as

$$\begin{aligned} R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}}T_{ACE\dot{F}} + \partial_{E\dot{F}}T_{ACD\dot{B}} + \varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} + \\ +T_{ACP\dot{B}}T_{QDF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}} - T_{ACP\dot{E}}T_{QFD\dot{B}}) + \\ +\varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}}\overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}}\overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0, \end{aligned} \tag{6.140}$$

where we have introduced the spinor indices in the matrices $R_{A\dot{B}C\dot{D}}$ and $T_{A\dot{B}}$ following the rule

$$\begin{aligned} R_{A\dot{B}C\dot{D}} \to R_{EFA\dot{B}C\dot{D}} = R_{EFkn}\sigma^k_{A\dot{B}}\sigma^n_{C\dot{D}}, \\ T_{A\dot{B}} \to T_{CDA\dot{B}} = T_{CDk}\sigma^k_{A\dot{B}}. \end{aligned} \tag{6.141}$$

Substituting into (6.140) the relationship (6.136), we will arrive at the equations (6.138).

Spintensors Ψ_{ABCE} and $\Phi_{AB\dot{C}\dot{E}}$ have the following notation for their components [38]:

$$\Psi_{ABCE} = \begin{array}{c|ccc} & & CE & \\ AB & 00 & 01 & 11 \\ \hline 00 & \Psi_0 & \Psi_1 & \Psi_2 \\ 01 & - & - & \Psi_3 \\ 11 & - & - & \Psi_4 \end{array}, \tag{6.142}$$

$$\Phi_{AB\dot{C}\dot{E}} = \begin{array}{c|ccc} & & \dot{C}\dot{E} & \\ AB & \dot{0}\dot{0} & \dot{0}\dot{1} & \dot{1}\dot{1} \\ \hline 00 & \Phi_{00} & \Phi_{01} & \Phi_{02} \\ 01 & \Phi_{10} & \Phi_{11} & \Phi_{12} \\ 11 & \Phi_{20} & \Phi_{21} & \Phi_{22} \end{array}, \tag{6.143}$$

$$\Lambda = \overline{\Lambda}. \tag{6.144}$$

Using the relationships (6.114), (6.115), (6.124), we can expand the equations (6.126) of the Newman-Penrose formalism component by component to arrive at the equations $(A.1)$ – $(A.8)$ plus the complex conjugate equations. Using the relationships (6.142)-(6.144) and (6.114) we also can expand the equations (6.138) of the Newman-Penrose formalism componentwise. We will thus end up with the equations $(B^{s+}.1)$–$(B^{s+}.18)$.

The spinor counterpart of the dual Riemann tensor

$$\overset{*}{R}_{ijkm}=\frac{1}{2}\varepsilon_{km}{}^{sp}R_{ijsp} \tag{6.145}$$

can be written as

$$\begin{aligned}\overset{*}{R}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}= i\,(\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}-\\ -\overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + \Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}}+\\ +2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}}))\,.\end{aligned} \tag{6.146}$$

It follows that

$$\begin{aligned}\overset{*}{R}_{A\dot{B}C\dot{D}EF}= \frac{1}{2}\varepsilon^{\dot{P}\dot{Q}}R_{A\dot{B}C\dot{D}E\dot{P}F\dot{Q}} = -i\,(-\varepsilon_{\dot{B}\dot{D}}\Psi_{ACEF}+\\ +\varepsilon_{EF}\Phi_{AC\dot{B}\dot{D}} + \Lambda\varepsilon_{\dot{B}\dot{D}}(\varepsilon_{AE}\varepsilon_{CF} + \varepsilon_{CE}\varepsilon_{AF}))\,,\end{aligned} \tag{6.147}$$

also

$$\begin{aligned}\overset{*}{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}}= \frac{1}{2}\varepsilon^{EF}R_{A\dot{B}C\dot{D}E\dot{P}F\dot{Q}} = i\left(\varepsilon_{\dot{A}\dot{C}}\overline{\Psi}_{\dot{B}\dot{D}\dot{P}\dot{Q}}-\right.\\ \left.-\varepsilon_{\dot{P}\dot{Q}}\overline{\Phi}_{AC\dot{B}\dot{D}} + \Lambda\varepsilon_{AC}(\varepsilon_{\dot{D}\dot{P}}\varepsilon_{\dot{B}\dot{Q}} + \varepsilon_{\dot{B}\dot{P}}\varepsilon_{\dot{D}\dot{Q}})\right).\end{aligned} \tag{6.148}$$

The dual Weyl tensor $\overset{*}{C}_{ijkm}$ corresponds to the spintensor of the form

$$\overset{*}{C}_{ijkm}\leftrightarrow\overset{*}{C}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}= i(\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}).$$

The self-dual spintensor $R_{A\dot{B}C\dot{D}EF}$ will be

$$R_{A\dot{B}C\dot{D}EF} = i\,\overset{*}{R}_{A\dot{B}C\dot{D}EF}= \Psi_{ACEF}\varepsilon_{\dot{B}\dot{D}}, \tag{6.149}$$

whereas the anti-self-dual tensor is

$$\overline{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}} = i\,\overset{*}{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}}= \varepsilon_{AC}\overline{\Psi}_{\dot{E}\dot{B}\dot{P}\dot{Q}}. \tag{6.150}$$

Proposition 6.11. The second Bianchi identities (D^{s+}) of the A_4 geometry in the spinor Δ-basis can be represented as

$$\begin{aligned}\frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{P\dot{X}}R_{ABG\dot{H}F\dot{E}} - \Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} -\\ -3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} +\\ +\Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} + \Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0,\end{aligned} \tag{6.151}$$

where

$$\varepsilon^{C\dot{D}F\dot{E}G\dot{H}R\dot{X}} = -i(\varepsilon^{CG}\varepsilon^{RF}\varepsilon^{\dot{D}\dot{E}}\varepsilon^{\dot{H}\dot{X}} - \varepsilon^{CF}\varepsilon^{GR}\varepsilon^{\dot{D}\dot{H}}\varepsilon^{\dot{E}\dot{X}}). \tag{6.152}$$

Proof. We will write the equations (6.79) as

$$\nabla^n \overset{*}{R}_{ACkn} - \overset{*}{R}_{EAkn}\, T_C{}^{En} - \overset{*}{R}_{AEkn}\, T_C{}^{En} = 0. \tag{6.153}$$

Multiplying these equations by $\sigma^k_{C\dot{D}}$ gives

$$\begin{aligned} \partial^{F\dot{E}}\, \overset{*}{R}_{BAC\dot{D}F\dot{E}} + \overset{*}{R}_{BAC\dot{D}F\dot{E}}\, \nabla^n \sigma_n{}^{F\dot{E}} + \\ + \overset{*}{R}_{BAR\dot{S}F\dot{E}}\, \sigma^k_{C\dot{D}}\, \partial^{F\dot{E}} \sigma_k{}^{R\dot{S}} - \\ - \overset{*}{R}_{BPC\dot{D}F\dot{E}}\, T_A{}^{PF\dot{E}} - \overset{*}{R}_{PAC\dot{D}F\dot{E}}\, T_B{}^{PF\dot{E}} = 0. \end{aligned} \tag{6.154}$$

Here we have used the relationships (6.94) and (6.133). Substituting into (6.154) the relationship (6.148), we will get

$$iD_{ABC\dot{D}} + A_{F\dot{E}P\dot{R}}{}^{i}\left(\sigma_{iC\dot{D}} R_{BA}{}^{P\dot{R}F\dot{E}} - 2\sigma_i{}^{P\dot{R}} R_{BAC\dot{D}}{}^{F\dot{E}}\right) = 0,$$

where $A_{F\dot{E}P\dot{R}}{}^{i}$ stands for the equations (6.125), rewritten as

$$\begin{aligned} A_{A\dot{B}C\dot{D}}{}^{i} = \partial_{A\dot{B}}\sigma^{i}{}_{C\dot{D}} - \partial_{C\dot{D}}\sigma^{i}{}_{A\dot{B}} - \\ -\varepsilon^{PQ}\left(T_{PAC\dot{D}}\sigma^{i}{}_{Q\dot{B}} - T_{PCA\dot{B}}\sigma^{i}{}_{Q\dot{D}}\right) - \\ -\varepsilon^{\dot{R}\dot{S}}\left(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^{i}{}_{A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^{i}{}_{C\dot{S}}\right) = 0, \end{aligned} \tag{6.155}$$

and $D_{ABC\dot{D}} = 0$ defines the equations (6.151)

$$\begin{aligned} D_{ABC\dot{D}} = \frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{R\dot{X}} R_{ABG\dot{H}F\dot{E}} - \\ -\Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} - 3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} + \\ +\Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} + \Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0, \end{aligned}$$

which proves the Proposition.

Proposition 6.12. The second Bianchi identities (6.151) of the A_4 geometry coincide with the Bianchi identities in the work by Newman-Penrose [40].

$$\begin{aligned} \partial^{P}{}_{\dot{D}}\Psi_{ABPC} - \partial^{X}{}_{(C}\Phi_{AB)\dot{D}\dot{X}} - 3\Psi_{PR(AB}T_{C)}{}^{PR}{}_{\dot{D}} - \\ -\Psi_{ABCP}T^{P}{}_{R}{}^{R}{}_{\dot{D}} + 2T^{P}{}_{(AB}{}^{\dot{X}}\Phi_{C)P\dot{X}\dot{D}} - \\ -\overline{T}_{\dot{X}\dot{D}\dot{V}(A}\Phi_{BC)}{}^{\dot{X}\dot{V}} - \overline{T}_{\dot{X}}{}^{\dot{V}}{}_{\dot{V}(A}\Phi_{BC)}{}^{\dot{X}}{}_{\dot{D}} = 0, \end{aligned} \tag{6.156}$$

$$3\partial_{A\dot{B}}\Lambda + \partial^{P\dot{X}}\Phi_{AP\dot{B}\dot{X}} - \varepsilon^{\dot{V}\dot{W}}\left(\Phi_{AP}{}^{\dot{X}}{}_{\dot{W}}\overline{T}_{\dot{B}\dot{X}\dot{V}}{}^{P} + \right.$$
$$\left. + \Phi_{AP\dot{B}}{}^{\dot{X}}\overline{T}_{\dot{X}\dot{W}\dot{V}}{}^{P}\right) + \Phi_{PR\dot{B}}{}^{\dot{X}}T_{A}{}^{PR}{}_{\dot{X}} +$$
$$+\Phi_{AP\dot{B}}{}^{\dot{X}}T^{P}{}_{R}{}^{R}{}_{\dot{X}} = 0 \qquad (6.157)$$

Proof. Using (6.147) and the equality

$$\overset{*}{R}_{ABC\dot{D}}{}^{R\dot{X}} = \frac{1}{2}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}R_{ABG\dot{H}F\dot{E}},$$

we find that in (6.151)

$$\frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{P\dot{X}}R_{ABG\dot{H}F\dot{E}} = \partial_{P\dot{X}}R_{ABC\dot{D}}{}^{R\dot{X}} =$$
$$= \partial_{P\dot{X}}\left(\varepsilon_{\dot{D}}{}^{\dot{X}}\Psi_{ABC}{}^{R} - \varepsilon_{AB}\Phi_{C}{}^{R}{}_{\dot{D}}{}^{\dot{X}} - \Lambda\varepsilon_{\dot{D}}{}^{\dot{X}}(\varepsilon_{CA}\varepsilon_{B}{}^{R} + \varepsilon_{BA}\varepsilon_{C}{}^{R})\right).$$

Substituting this relationship into (6.151) gives

$$\partial^{P}{}_{\dot{D}}\Psi_{ABPC} - \partial_{C}{}^{\dot{X}}\Phi_{AB\dot{D}\dot{X}} + 2\varepsilon_{C(A}\partial_{B)\dot{D}}\Lambda - \Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} -$$
$$-3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} + \Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} +$$
$$+\Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0. \qquad (6.158)$$

The part of (6.158) symmetrical in the indices C and B can be written as (6.156); and the part skew-symmetrical in these indices looks like (6.157).

By writing the second Bianchi identities (D^{s+}) of the A_4 geometry component by component, we obtain [40]

$$(D - 4\rho - 2\varepsilon)\Psi_1 - (\overline{\delta} - 4\alpha + \pi)\Psi_0 +$$
$$+3\kappa\Psi_2 + (\delta - 2\beta - 2\overline{\alpha} + \overline{\pi})\Phi_{00} -$$
$$-(D - 2\overline{\rho} - 2\varepsilon)\Phi_{01} - 2\kappa\Phi_{11} +$$
$$+2\sigma\Phi_{10} - \overline{\kappa}\Phi_{02} = 0, \qquad (D^{s+}.1)$$

$$(D - 3\rho)\Psi_2 - (\overline{\delta} + 2\pi - 2\alpha)\Psi_1 +$$
$$+2\kappa\Psi_3 + \lambda\Psi_0 + (\delta - 2\overline{\alpha} +$$
$$+\overline{\pi})\Phi_{10} - (D - 2\overline{\rho})\Phi_{11} - \kappa\Phi_{21} -$$
$$-\overline{\kappa}\Phi_{12} - \mu\Phi_{00} + \pi\Phi_{01} + \sigma\Phi_{20} - D\Lambda = 0, \qquad (D^{s+}.2)$$

$$(D - 2\rho + 2\varepsilon)\Psi_3 - (\overline{\delta} + 3\pi)\Psi_2 +$$
$$+2\lambda\Psi_1 + \kappa\Psi_4 + (\delta - 2\overline{\alpha} + 2\beta +$$
$$+\overline{\pi})\Phi_{20} - (D - 2\overline{\rho} + 2\varepsilon)\Phi_{21} -$$
$$-2\mu\Phi_{10} + 2\pi\Phi_{11} - \overline{\kappa}\Phi_{22} - 2\delta\Lambda = 0, \qquad (D^{s+}.3)$$

$$
\begin{aligned}
&(\delta - 4\tau - 2\beta)\Psi_1 - (\Delta - 4\gamma + \mu)\Psi_0 + \\
&+3\sigma\Psi_2 + (\delta - 2\beta + 2\overline{\pi})\Phi_{01} - (D - 2\varepsilon + 2\overline{\varepsilon} - \\
&-\overline{\rho})\Phi_{02} - 2\kappa\Phi_{12} + 2\sigma\Phi_{11} - \\
&-\overline{\lambda}\Phi_{00} = 0, \qquad (D^{s+}.4)
\end{aligned}
$$

$$
\begin{aligned}
&(\delta - 3\tau)\Psi_2 - (\Delta + 2\mu - 2\gamma)\Psi_1 + 2\sigma\Psi_3 + \\
&+\nu\Psi_0 + (\delta + 2\overline{\pi})\Phi_{11} - (D + 2\overline{\varepsilon} - \\
&-\overline{\rho})\Phi_{12} - \kappa\Phi_{22} - \mu\Phi_{01} + \pi\Phi_{02} + \\
&+\sigma\Phi_{21} - \overline{\lambda}\Phi_{10} - \delta\Lambda = 0, \qquad (D^{s+}.5)
\end{aligned}
$$

$$
\begin{aligned}
&(\delta + 2\beta - 2\tau)\Psi_3 - (\Delta + 3\mu)\Psi_2 + 2\nu\Psi_1 + \\
&+\sigma\Psi_4 + (\delta + 2\beta + 2\overline{\pi})\Phi_{21} - \\
&-(D + 2\varepsilon + 2\overline{\varepsilon} - \overline{\rho})\Phi_{22} - \\
&-2\mu\Phi_{11} + 2\pi\Phi_{12} - \overline{\lambda}\Phi_{20} - 2\Delta\Lambda = 0, \quad (D^{s+}.6)
\end{aligned}
$$

$$
\begin{aligned}
&(D + 4\varepsilon - \rho)\Psi_4 - (\overline{\delta} + 4\pi + 2\alpha)\Psi_3 + \\
&+3\lambda\Psi_2 + (\Delta + 2\gamma - 2\overline{\gamma} + \overline{\mu})\Phi_{20} - \\
&-(\overline{\delta} + 2\alpha - 2\overline{\tau})\Phi_{21} - 2\nu\Phi_{10} + \\
&+2\lambda\Phi_{11} - \overline{\sigma}\Phi_{22} = 0, \qquad (D^{s+}.7)
\end{aligned}
$$

$$
\begin{aligned}
&(\delta + 4\beta - \tau)\Psi_4 - (\Delta + 2\gamma + 4\mu)\Psi_3 + 3\nu\Psi_2 + \\
&+(\Delta + 2\gamma + 2\overline{\mu})\Phi_{21} - (\overline{\delta} + 2\alpha + \\
&+2\overline{\beta} - \overline{\tau})\Phi_{22} - 2\nu\Phi_{11} + 2\lambda\Phi_{12} - \\
&-\overline{\nu}\Phi_{20} = 0, \qquad (D^{s+}.8)
\end{aligned}
$$

$$
\begin{aligned}
&(D - 2\rho - 2\overline{\rho})\Phi_{11} - (\delta - 2\overline{\alpha} - 2\tau + \\
&+\overline{\pi})\Phi_{10} - (\delta - 2\overline{\tau} - 2\alpha + \pi)\Phi_{01} + \\
&+(\Delta + 2\gamma - 2\overline{\gamma} + \mu + \overline{\mu})\Phi_{00} + \\
&+\overline{\kappa}\Phi_{12} + \kappa\Phi_{21} - \overline{\sigma}\Phi_{02} - \\
&-\sigma\Phi_{20} + 3D\Lambda = 0, \qquad (D^{s+}.9)
\end{aligned}
$$

$$
\begin{aligned}
&(D - 2\rho + 2\overline{\varepsilon} - \overline{\rho})\Phi_{12} - \\
&-(\delta + 2\overline{\pi} - 2\tau)\Phi_{11} - (\overline{\delta} + 2\overline{\beta} - \\
&-2\alpha - \overline{\tau} + \pi)\Phi_{02} + (\Delta + 2\overline{\mu} - 2\gamma + \\
&+\mu)\Phi_{01} + \kappa\Phi_{22} - \overline{\nu}\Phi_{00} - \\
&-\overline{\lambda}\Phi_{10} - \sigma\Phi_{21} + 3\delta\Lambda = 0, \qquad (D^{s+}.10)
\end{aligned}
$$

$$(D + 2\varepsilon + 2\overline{\varepsilon} - \rho - \overline{\rho})\Phi_{22} -$$
$$-(\delta + 2\overline{\pi} + 2\beta - \tau)\Phi_{21} - (\overline{\delta} +$$
$$+2\overline{\beta} + 2\pi - \overline{\tau})\Phi_{12} + (\Delta + 2\mu +$$
$$+2\overline{\mu})\Phi_{11} - \overline{\nu}\Phi_{10} - \nu\Phi_{01} +$$
$$+\overline{\lambda}\Phi_{20} + \lambda\Phi_{02} + 3\Delta\Lambda = 0. \qquad (D^{s+}.11)$$

To arrive at the complete set of the second Bianchi (D) identities of the A_4 geometry, we will have to add to these equations their complex conjugate (D^{s-}).

6.7 Variational principle of derivation of the structural Cartan equations and the second Bianchi identities of A$_4$ geometry

To begin with, we will consider the derivation of the structural equations (B) and of the second Bianchi identities (D) for self-dual and anti-self-dual fields of Riemannian curvature, whose Carmeli matrices obey the conditions

$$R_{kn} = \pm i \overset{*}{R}_{kn} ,$$

$$R^{+}_{kn} = \pm i \overset{*}{R}_{kn} ,$$

where

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0,$$
$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_k, T^{+}_n] = 0,$$

and

$$\overset{*}{R}_{kn} = \frac{1}{2}\varepsilon^{knps} R_{ps} ,$$
$$\overset{*}{R}{}^{+}_{kn} = \frac{1}{2}\varepsilon^{knps} R^{+}_{ps} .$$

Let us take the Lagrange function in the form

$$L_1 = -\frac{1}{4}(-g)^{1/2} Tr(R_{kn}R^{kn}) + \text{complex conjugate part.} \qquad (6.159)$$

Varying this expression in T_k and T^{+}_k, we will arrive at the equations (D)

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \qquad (6.160)$$

$$\nabla^n \overset{*}{R}{}^{+}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \qquad (6.161)$$

For arbitrary fields of Riemannian curvature the Lagrange function looks like

$$L_2 = -\frac{1}{2}(-g)^{1/2} Tr\left(\overset{*}{R}{}^{kn}(-\frac{1}{2}R_{kn} - 2\nabla_{[k}T_{n]} + [T_k, T_n])\right) + \text{c.c. part.} \quad (6.162)$$

Variation of this Lagrangian in $\overset{*}{R}_{kn}$ and $\overset{*}{R}{}^{+}{}_{kn}$ yields the second Bianchi identities (D)

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \qquad (D^{s+})$$

$$\nabla^n \overset{*}{R}{}^{+}{}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \qquad (D^{s-})$$

On the other hand, variation of the Lagrangian (6.162) in T_k and $T^{+}{}_k$ gives the second structural Cartan equations (B) of the A_4 geometry

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0, \qquad (B^{s+})$$

$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_k, T^{+}_n] = 0, \qquad (B^{s-})$$

and

$$\overset{*}{R}_{kn} = \frac{1}{2}\varepsilon^{knps}R_{ps},$$

$$\overset{*}{R}{}^{+}_{kn} = \frac{1}{2}\varepsilon^{knps}R^{+}_{ps}.$$

Independent variables in the Lagrangian (6.162) are the quantities R_{kn}, R^{+}_{kn}, T_k, and T^{+}_k. To obtain from them using the variational principle, the first structural Cartan equations (A) of the A_4 geometry

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0, \qquad (A^s)$$

we will have to introduce into the Lagrangian (6.162) as independent variables the matrices σ^i. This can be done by modifying the Lagrangian (6.162) as it has been done in [49].

We now write the equations (A), (B) and (D) in spinor form :

$$(A) \qquad A^{i}{}_{A\dot{B}C\dot{D}} = 0, \qquad (6.163)$$

$$(B) \quad B_{F\dot{E}AC D\dot{B}} = 0 + \text{c.c. equations}, \qquad (6.164)$$

$$(D) \quad D_{ABC\dot{D}} = 0 + \text{c.c. equations}, \qquad (6.165)$$

where

$$A^{i}_{A\dot{B}C\dot{D}} = \partial_{A\dot{B}}\sigma^{i}_{C\dot{D}} - \partial_{C\dot{D}}\sigma^{i}_{A\dot{B}} - \varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^{i}_{Q\dot{B}} - T_{PCA\dot{B}}\sigma^{i}_{Q\dot{D}}) - \varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^{i}_{A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^{i}_{C\dot{S}}) = 0, \qquad (6.166)$$

$$B_{ACF\dot{E}D\dot{B}} = R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}}T_{ACE\dot{F}} + \partial_{E\dot{F}}T_{ACD\dot{B}} + \varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} + T_{ACP\dot{B}}T_{QDF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}} - T_{ACP\dot{E}}T_{QFD\dot{B}}) + \varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}}\overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}}\overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0, \qquad (6.167)$$

$$D_{ABC\dot{D}} = \frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{R\dot{X}}R_{ABG\dot{H}F\dot{E}} -$$
$$-\Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} - 3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} +$$
$$+\Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} + \Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} + \Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0 \qquad (6.168)$$

and consider the Lagrangian

$$L_3 = \overset{*}{R}{}^{B}{}_{\dot{Q}}{}^{A\dot{Q}kn}\left((2\nabla_n T_{ABk} + 2T_{PAn}T_B{}^P{}_k) - \frac{1}{4}R_{BPA}{}^{P}{}_{nk}\right) + \text{c.c. part.} \qquad (6.169)$$

Here $\overset{*}{R}{}^{B}{}_{\dot{Q}}{}^{A\dot{Q}kn} = \varepsilon^{nkjm}R^{B}{}_{\dot{Q}}{}^{A\dot{Q}}{}_{jm}$ and ε^{nkjm} is a completely skew-symmetrical Levi-Chivita symbol.

If we take $R^{B}{}_{\dot{Q}}{}^{A\dot{Q}kn}$ and T_{PAn} to be independent variables and use the conventional variational procedure, we will obtain the following equations:

$$(B^{s}+) \quad \frac{1}{2}R_{B\dot{P}A}{}^{\dot{P}}{}_{kn} - 2\nabla_{[k}T_{|AB|n]} + 2T_{PA[k}T^{P}{}_{|B|n]} = 0, \qquad (6.170)$$

$$(B^{s}-) \qquad \text{complex conjugate equations,} \qquad (6.171)$$

$$(D^{s}+) \qquad \nabla^{k}\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{nk} - 2\overset{*}{R}_{P\dot{Q}(A}{}^{\dot{Q}}{}_{|nk|}T^{Pk}_{B)} = 0, \qquad (6.172)$$

$$(D^{s}-) \qquad \text{complex conjugate equations.} \qquad (6.173)$$

Multiplying equations (6.170) by $\sigma_{C\dot{D}}{}^{n}\sigma_{F\dot{E}}{}^{n}$ gives

$$\partial_{F\dot{E}}T_{ABC\dot{D}} - \partial_{C\dot{D}}T_{ABF\dot{E}} + T_{PAF\dot{E}}T^{P}_{BA\dot{D}} - T_{PAC\dot{D}}T^{P}_{BF\dot{E}} -$$
$$-\frac{1}{2}R_{B\dot{Q}A}{}^{P}{}_{F\dot{E}C\dot{D}} + T_{ABn}(\partial_{C\dot{D}}\sigma_{F\dot{E}}{}^{n} - \partial_{F\dot{E}}\sigma_{C\dot{D}}{}^{n}) = 0. \qquad (6.174)$$

Using the notation (6.166) and (6.167), we will write (6.174) as

$$B_{ACF\dot{E}D\dot{B}} + A^{n}{}_{CDF\dot{E}}T_{ABn} = 0. \qquad (6.175)$$

We will now multiply (6.172) by $\sigma_{C\dot{D}}{}^{k}$ to get the relationship

$$\partial^{F\dot{E}}\overset{*}{R}{}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}} + \overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}\nabla^{k}\sigma_{k}{}^{F\dot{E}} +$$
$$+\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{R\dot{S}E\dot{F}}\sigma^{n}{}_{C\dot{D}}\partial^{F\dot{E}}\sigma^{R\dot{S}}{}_{n} -$$
$$-\overset{*}{R}_{B\dot{Q}P}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}T_{A}{}^{PF\dot{E}} - \overset{*}{R}_{P\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}T_{B}{}^{PF\dot{E}} = 0 \qquad (6.176)$$

or, from (6.166) and (6.167),

$$iD_{ABC\dot{D}} + A_{F\dot{E}P\dot{R}}{}^{n}\left(\frac{1}{2}\sigma_{nC\dot{D}}\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}P\dot{R}E\dot{F}} - \sigma_{n}{}^{P\dot{R}}\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}}{}^{E\dot{F}}\right) = 0. \qquad (6.177)$$

Here we have also used the relationship

$$\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}} = i\left(2\Psi_{BACF}\varepsilon_{\dot{D}\dot{E}} - 2\varepsilon_{CF}\Phi_{AB\dot{D}\dot{E}} + 2\Lambda\varepsilon_{\dot{D}\dot{E}}(\varepsilon_{BF}\varepsilon_{AC} + \right.$$
$$\left. + \varepsilon_{BC}\varepsilon_{AF})\right).$$

It is clear that from the Lagrangian (6.169) it is impossible to obtain the first structural Cartan equations (A) of the A_4 geometry, since it does not contain $\sigma_{C\dot{D}}{}^{n}$.

Let us add to the Lagrangian (6.169) the term

$$\lambda_j^{A\dot{B}C\dot{D}} A^j{}_{A\dot{B}C\dot{D}} \tag{6.178}$$

where the quantities $\lambda^{A\dot{B}C\dot{D}}$ play the role of Lagrange factors

$$L_4 = L_3 + \lambda_j^{A\dot{B}C\dot{D}} A^j{}_{A\dot{B}C\dot{D}} + \text{c.c. part.} \tag{6.179}$$

The quantities $\lambda_j^{A\dot{B}C\dot{D}}$, just like $A^j{}_{A\dot{B}C\dot{D}}$, are Hermitian matrices, which are skew-symmetrical in the pair of indices [49] $A\dot{B}$ and $C\dot{D}$. Varying the Lagrange density (6.179) in $\sigma_{C\dot{D}}{}^{n}$ gives [49]

$$A^j{}_{A\dot{B}C\dot{D}} = 0 \tag{6.180}$$

and

$$D_{ABC\dot{D}} = \sigma_k^{P\dot{R}}\sigma^{\dot{X}}_{nB}(\lambda^n_{A\dot{X}P\dot{R}} - \overline{\lambda}^n_{A\dot{X}P\dot{R}})\sigma^k_{C\dot{D}} = 0. \tag{6.181}$$

Since $\lambda^n_{A\dot{X}P\dot{R}}$ are Hermitian matrices, from (6.181) we have the equations (D^{s+})

$$D_{ABC\dot{D}} = 0. \tag{6.182}$$

Hence varying the complex conjugate part of the Lagrangian (6.179) gives

$$\overline{D}_{\dot{A}\dot{B}\dot{C}D} = 0. \tag{6.183}$$

and of the Lagrangian (6.179) in $R^{B}{}_{\dot{Q}A}{}^{\dot{Q}kn}$ gives

$$B_{ACF\dot{E}D\dot{B}} + A^{n}{}_{CDF\dot{E}}T_{ABn} = 0 \tag{6.184}$$

or, from (6.180),

$$B_{ACF\dot{E}D\dot{B}} = 0. \tag{6.185}$$

Variation of the complex conjugate part in $\overline{R}^{\dot{B}}{}_{Q}{}^{\dot{A}Qkn}$ yields

$$\overline{B}_{ACF\dot{E}D\dot{B}} = 0. \tag{6.186}$$

It has thus been shown that from the Lagrangian (6.179) follow the first and second of the structural Cartan equations of the A_4 geometry (equations (6.180), (6.185) and (6.186)), and also the second of the Bianchi identities (equations (6.182) and (6.183)).

6.8 Decomposition of spinor fields of A_4 geometry into irreducible parts

The torsion tensor $\Omega_{jk}^{..i}$ of the A_4 space has 24 independent components, and it can be represented as the sum of three irreducible parts as follows:

$$\Omega^{i}_{.jk} = \frac{2}{3}\delta^{i}_{[k}\Omega_{j]} + \frac{1}{3}\varepsilon^{n}_{jks}\hat{\Omega}^{s} + \overline{\Omega}^{i}_{.jk}, \tag{6.187}$$

where

$$\Omega^{i}_{.jk} = g^{im}g_{ks}\Omega_{mj}^{..s}, \tag{6.188}$$

and the vector Ω_j, pseudovector $\hat{\Omega}_j$ and the traceless part of the torsion $\overline{\Omega}^{i}_{.jk}$ are given by

$$\Omega_j = \Omega^{i}_{.ji}, \tag{6.189}$$

$$\hat{\Omega}_j = \frac{1}{2}\varepsilon_{jins}\Omega^{ins}, \tag{6.190}$$

$$\overline{\Omega}^{s}_{.js} = 0, \quad \overline{\Omega}_{ijs} + \overline{\Omega}_{jsi} + \overline{\Omega}_{sij} = 0. \tag{6.191}$$

In the spinor basis the spinor representation of the Ricci rotation coefficients $T_{ABC\dot{C}}$ has the form [40]

$$T_{ABC\dot{C}} = \frac{1}{2}\left(A_{ABC\dot{C}} + \frac{1}{3}(\varepsilon_{AC}\alpha_{B\dot{C}} + \varepsilon_{BC}\alpha_{A\dot{C}})\right), \tag{6.192}$$

where the spinor $A_{ABC\dot{C}}$ is completely symmetrical in the unprimed indices

$$A_{ABC\dot{C}} = A_{(ABC)\dot{C}}, \tag{6.193}$$

and the spinor $\alpha_{B\dot{C}}$ is given by

$$\alpha_{A\dot{C}} = A_{AB}{}^{B}{}_{\dot{C}}. \tag{6.194}$$

In turn, the spinor $\alpha_{A\dot{C}}$ can be decomposed into the Hermitian and anti-Hermitian parts:

$$\alpha_{A\dot{C}} = \kappa_{A\dot{C}} - i\mu_{A\dot{C}}, \tag{6.195}$$

where

$$\kappa_{A\dot{C}} = \frac{1}{2}(\alpha_{A\dot{C}} + \overline{\alpha}_{\dot{A}C}), \quad \mu_{A\dot{C}} = \frac{1}{2}i(\alpha_{A\dot{C}} - \overline{\alpha}_{\dot{A}C}) \tag{6.196}$$

and

$$\overline{\kappa_{A\dot{C}}} = \overline{\kappa}_{\dot{A}C} = \kappa_{C\dot{A}}, \quad \overline{\mu_{A\dot{C}}} = \overline{\mu}_{A\dot{C}} = \mu_{C\dot{A}}. \tag{6.197}$$

The irreducible parts of torsion (6.189)-(6.191) and the spinors (6.193)-(6.197) are related by

$$\Omega_j \longleftrightarrow \kappa_{A\dot{C}}, \tag{6.198}$$

$$\hat{\Omega}_j \longleftrightarrow \mu_{A\dot{C}}, \tag{6.199}$$

$$\overline{\Omega}^{k}_{.js} \longleftrightarrow A_{ABC\dot{C}}. \qquad (6.200)$$

Since

$$\Omega_{ijk} = g_{sk}\Omega_{ij}^{..s}, \qquad (6.201)$$

we have

$$\Omega_{A\dot{A}B\dot{B}C\dot{C}} \longleftrightarrow \Omega_{ijk}, \qquad (6.202)$$

$$\Omega_{A\dot{A}B\dot{B}C\dot{C}} = \frac{1}{2}(\Omega_{ABC\dot{C}}\varepsilon_{\dot{A}\dot{B}} + \overline{\Omega}_{\dot{A}\dot{B}\dot{C}C}\varepsilon_{AB}), \qquad (6.203)$$

$$\Omega_{ABC\dot{C}} = A_{C(AB)\dot{C}} + \overline{\alpha}_{\dot{C}(A}\varepsilon_{B)C}. \qquad (6.204)$$

By definition, the spinor $A_{ABC\dot{C}}$ is transformed in the $D(3/2.1/2)$ irreducible representation of the group $SL(2,C)$. Consequently, the spinors $\kappa_{A\dot{C}}$ and $\mu_{A\dot{C}}$ are transformed in the $D(1/2.1/2)$ irreducible representation of the group $SL(2.C)$. Using the relationship (6.124), we can find the components of the spinors $\kappa_{A\dot{C}}$ and $\mu_{A\dot{C}}$ [50]

$$\kappa_{A\dot{C}} = \begin{pmatrix} \frac{1}{2}(\rho+\overline{\rho}) - \frac{1}{2}(\varepsilon+\overline{\varepsilon}) & \frac{1}{2}(\tau+\beta) + \frac{1}{2}(\overline{\alpha}-\overline{\pi}) \\ \frac{1}{2}(\overline{\tau}-\overline{\beta}) + \frac{1}{2}(\alpha-\pi) & \frac{1}{2}(\gamma+\overline{\gamma}) - \frac{1}{2}(\mu+\overline{\mu}) \end{pmatrix}, \qquad (6.205)$$

$$\mu_{A\dot{C}} = i\begin{pmatrix} \frac{1}{2}(\rho-\overline{\rho}) - \frac{1}{2}(\varepsilon-\overline{\varepsilon}) & \frac{1}{2}(\tau-\beta) - \frac{1}{2}(\overline{\alpha}-\overline{\pi}) \\ -\frac{1}{2}(\overline{\tau}-\overline{\beta}) + \frac{1}{2}(\alpha-\pi) & \frac{1}{2}(\gamma-\overline{\gamma}) - \frac{1}{2}(\mu-\overline{\mu}) \end{pmatrix}. \qquad (6.206)$$

The Riemann tensor represented in terms of irreducible parts is

$$R_{ijkm} = C_{ijkm} + g_{i[k}R_{m]j} + g_{j[k}R_{m]i} + \frac{1}{3}Rg_{i[m}g_{k]j}. \qquad (6.207)$$

In the spinor basis this becomes [40]

$$\begin{aligned} R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \\ +\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}} + \\ +\overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \\ +\varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}). \end{aligned} \qquad (6.208)$$

We also have the following connection:

$$\begin{aligned} C_{ijkm} \longleftrightarrow \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \\ R_{ij} \longleftrightarrow 2\Phi_{AB\dot{C}\dot{D}} + 6\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}, \\ R \longleftrightarrow 24\Lambda, \end{aligned} \qquad (6.209)$$

where the spinors Ψ_{ABCD}, $\Phi_{AB\dot{C}\dot{D}}$ and Λ meet the following symmetry conditions:

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})}, \quad \Lambda = \overline{\Lambda}$$

and belong to the $D(2.0)$, $D(1.1)$ and $D(0.0)$ irreducible representations of the group $SL(2.C)$, respectively.

6.9 Spinor set of Einstein-Yang-Mills equations

In the first part of the book it was shown that the structural Cartan equations of the geometry of absolute parallelism (A) and (B) can be represented as an extended set of Einstein-Yang-Mills equations

$$\boxed{\begin{array}{lr} \nabla_{[k}e^{a}{}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (A) \\ R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, & (B.1) \\ C^{i}{}_{jkm} + 2\nabla_{[k}T^{i}{}_{|j|m]} + 2T^{i}_{s[k}T^{s}{}_{|j|m]} = -\nu J^{i}{}_{jkm}. & (B.2) \end{array}} \tag{6.210}$$

We will write this set of equations in the spinor basis. To this end, we will make use of the Carmeli matrices and the Newman-Penrose spinor formalism. Suppose now we have the right spin A_4 geometry, then its equations (A) and (B) have the form

$$\partial_{C\dot{D}}\sigma^{i}{}_{A\dot{B}} - \partial_{A\dot{B}}\sigma^{i}{}_{C\dot{D}} = (T_{C\dot{D}})^{P}_{A}\sigma^{i}{}_{P\dot{B}} +$$
$$+\sigma^{i}{}_{A\dot{R}}(T^{+}_{\dot{D}C})^{\dot{R}}_{\dot{B}} - (T_{A\dot{B}})^{P}_{C}\sigma^{i}{}_{P\dot{D}} - \sigma^{i}{}_{C\dot{R}}(T^{+}_{\dot{B}A})^{\dot{R}}_{\dot{D}}, \qquad (A^{s+})$$

$$R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}}T_{A\dot{B}} - \partial_{A\dot{B}}T_{C\dot{D}} - (T_{C\dot{D}})^{F}_{A}T_{F\dot{B}} -$$
$$-(T^{+}_{\dot{D}C})^{\dot{F}}_{\dot{B}}T_{A\dot{F}} + (T_{A\dot{B}})^{F}_{C}T_{F\dot{D}} + (T^{+}_{\dot{B}A})^{\dot{F}}_{\dot{D}}T_{C\dot{F}} + [T_{A\dot{B}}, T_{C\dot{D}}], \qquad (B^{s+})$$

where the components of the matrices $\sigma^{i}{}_{A\dot{B}}$, $T_{A\dot{B}}$ and $R_{A\dot{B}C\dot{D}}$ are given by (6.115), (6.88) and (6.103), respectively.

Proposition 6.13. Equations $(B.1)$ in the spinor basis are

$$2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}. \tag{6.211}$$

Proof. In terms of the irreducible spinors (6.209) $P - Q$ the components of the spinor matrices $R_{A\dot{B}C\dot{D}}$ are given by [51]

$$(R_{A\dot{B}C\dot{D}})_{P}{}^{Q} = \varepsilon_{\dot{D}\dot{B}}\left(\Psi_{CAP}{}^{Q} - \Lambda(\varepsilon_{PC}\delta^{Q}_{A} + \varepsilon_{PA}\delta^{Q}_{C})\right) + \varepsilon_{CA}\Phi_{P}{}^{Q}{}_{\dot{D}\dot{B}}, \tag{6.212}$$

where

$$(C_{A\dot{B}C\dot{D}})_{P}{}^{Q} = \varepsilon_{\dot{D}\dot{B}}\Psi_{CAP}{}^{Q} \tag{6.213}$$

are the $P - Q$ components of the spinor matrices of the Weyl tensor with the the components

$$C_{0\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 & -\Psi_1 \end{pmatrix}, \quad C_{1\dot{1}1\dot{0}} = \begin{pmatrix} \Psi_3 & -\Psi_2 \\ \Psi_4 & -\Psi_3 \end{pmatrix},$$
$$C_{1\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_2 & -\Psi_1 \\ \Psi_3 & -\Psi_2 \end{pmatrix}, \quad C_{1\dot{0}0\dot{1}} = \begin{pmatrix} -\Psi_2 & \Psi_1 \\ -\Psi_3 & \Psi_2 \end{pmatrix}, \tag{6.214}$$

and related with the spinor $\Lambda\varepsilon_{\dot{D}\dot{B}}(\varepsilon_{PC}\delta^{A}{}_{Q} + \varepsilon_{PA}\delta_{C}{}^{Q})$ and $\varepsilon_{CA}\Phi_{P}{}^{Q}{}_{\dot{D}\dot{B}}$ are the trace and traceless parts of the Ricci tensor

$$\Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = -\frac{1}{4}\sigma^{k}{}_{A\dot{C}}\sigma^{n}{}_{B\dot{D}}Rg_{kn}, \tag{6.215}$$

$$\Phi_{AB\dot{C}\dot{D}} = \frac{1}{2}\sigma^{k}{}_{A\dot{C}}\sigma^{n}{}_{B\dot{D}}\left(R_{kn} - \frac{1}{4}g_{kn}R\right). \tag{6.216}$$

Substituting relationships (6.215) and (6.216) into (6.211) and multiplying the resultant expression by $\sigma^{A\dot{C}}{}_{k}\sigma^{B\dot{D}}{}_{n}$, we arrive at the equations ($\boldsymbol{B}$.1).

We now represent the matrix $R_{A\dot{B}C\dot{D}}$ as the sum

$$R_{A\dot{B}C\dot{D}} = C_{A\dot{B}C\dot{D}} + \nu J_{A\dot{B}C\dot{D}}, \tag{6.217}$$

where the matrix current $J_{A\dot{B}C\dot{D}}$ has the components [52]:

$$J_{0\dot{1}0\dot{0}} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \frac{1}{6}T & 0 \end{pmatrix}, \quad J_{1\dot{1}1\dot{0}} = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{6}T \\ 0 & 0 \end{pmatrix},$$

$$J_{1\dot{0}0\dot{0}} = \frac{1}{2}\begin{pmatrix} T_{1\dot{0}0\dot{0}} & -T_{0\dot{0}0\dot{0}} \\ T_{1\dot{0}1\dot{0}} & -T_{1\dot{0}0\dot{0}} \end{pmatrix},$$

$$J_{1\dot{1}0\dot{1}} = \frac{1}{2}\begin{pmatrix} T_{0\dot{1}1\dot{1}} & -T_{0\dot{1}0\dot{1}} \\ T_{1\dot{1}1\dot{1}} & -T_{0\dot{1}1\dot{1}} \end{pmatrix}, \tag{6.218}$$

$$J_{1\dot{1}0\dot{0}} = \frac{1}{2}\begin{pmatrix} T_{1\dot{1}0\dot{0}} & -T_{0\dot{1}0\dot{0}} \\ T_{1\dot{0}1\dot{1}} & -T_{1\dot{1}0\dot{0}} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{1}{6}T & 0 \\ 0 & -\frac{1}{6}T \end{pmatrix},$$

$$J_{1\dot{0}0\dot{1}} = \frac{1}{2}\begin{pmatrix} T_{1\dot{1}0\dot{0}} & -T_{0\dot{1}0\dot{0}} \\ T_{1\dot{0}1\dot{1}} & -T_{1\dot{1}0\dot{0}} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\frac{1}{6}T & 0 \\ 0 & \frac{1}{6}T \end{pmatrix}.$$

Here

$$T_{A\dot{B}C\dot{D}} = \sigma^{k}{}_{A\dot{C}}\sigma^{n}{}_{B\dot{D}}T_{kn}, \tag{6.219}$$

$$T = g^{jm}T_{jm}, \tag{6.220}$$

and the energy-momentum tensor T_{kn} is given in terms of the Ricci rotation coefficients by

$$T_{jm} = -\frac{2}{\nu}\Big\{\nabla_{[i}T^{i}{}_{|j|m]} + T^{i}{}_{s[i}T^{s}{}_{|j|m]} -$$
$$-\frac{1}{2}g^{pn}g_{jm}\left(\nabla_{[i|}T^{i}{}_{|p|n]} + T^{i}{}_{s[i}T^{s}{}_{|p|n]}\right)\Big\}. \tag{6.221}$$

In the special case where the field $T^{i}{}_{jk}$ is skew-symmetric in all the three indices, the tensor (6.219) is [33]

$$T_{jm} = \frac{1}{\nu}\left(\hat{\Omega}_{j}\hat{\Omega}_{m} - \frac{1}{2}g_{jm}\hat{\Omega}^{i}\hat{\Omega}_{i}\right). \tag{6.222}$$

Multiplying this by $\sigma^{j}{}_{A\dot{C}}\sigma^{m}{}_{B\dot{D}}$ and using (6.199), we get

$$T_{A\dot{B}C\dot{D}} = \frac{1}{\nu}\left(\mu_{A\dot{B}}\mu_{C\dot{D}} - \frac{1}{2}\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\mu_{P\dot{Q}}\mu^{P\dot{Q}}\right). \tag{6.223}$$

In addition, we obtain

$$T = g^{jm}T_{jm} = -\frac{1}{\nu}\hat{\Omega}_j\hat{\Omega}^j = -\frac{1}{\nu}\mu_{P\dot{Q}}\mu^{P\dot{Q}}. \qquad (6.224)$$

Hence the "density of spinor matter" is

$$\rho = -\frac{1}{\nu c^2}\mu_{P\dot{Q}}\mu^{P\dot{Q}}. \qquad (6.225)$$

We substitute (6.217) into the spinor equations (B^{s+}) go get

$$2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}, \qquad (B^{s+}.1)$$

$$C_{A\dot{B}C\dot{D}} - \partial_{C\dot{D}}T_{A\dot{B}} + \partial_{A\dot{B}}T_{C\dot{D}} + (T_{C\dot{D}})_A^F T_{F\dot{B}} + (T^+_{\dot{D}C})_{\dot{B}}^{\dot{F}} T_{A\dot{F}} -$$

$$-(T_{A\dot{B}})_C^F T_{F\dot{D}} - (T^+_{\dot{B}A})_{\dot{D}}^{\dot{F}} T_{C\dot{F}} - [T_{A\dot{B}}, T_{C\dot{D}}] = -\nu J_{A\dot{B}C\dot{D}}. \qquad (B^{s+}.2)$$

To conclude, we will write the extended set of Einstein-Yang-Mills equations as

$$\boxed{\begin{array}{lr}
\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})_{\dot{B}}^{\dot{R}} - & \\
-(T_{A\dot{B}})_C^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})_{\dot{D}}^{\dot{R}}, & (A^s) \\
2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}, & (B^{s+}.1) \\
C_{A\dot{B}C\dot{D}} - \partial_{C\dot{D}}T_{A\dot{B}} + \partial_{A\dot{B}}T_{C\dot{D}} + (T_{C\dot{D}})_A^F T_{F\dot{B}} + (T^+_{\dot{D}C})_{\dot{B}}^{\dot{F}} T_{A\dot{F}} - & \\
-(T_{A\dot{B}})_C^F T_{F\dot{D}} - (T^+_{\dot{B}A})_{\dot{D}}^{\dot{F}} T_{C\dot{F}} - [T_{A\dot{B}}, T_{C\dot{D}}] = -\nu J_{A\dot{B}C\dot{D}}. & (B^{s+}.2)
\end{array}}$$

where the spinor indices take on the values $A, B, D \ldots = 0, 1$, $\dot{A}, \dot{B}, \dot{D} \ldots = \dot{0}, \dot{1}$.

6.10 Formalism of two-component spinors

We will introduce the two-component spinors o^α and ι^α [53], connected with the components of the spinor dyad ξ^α_β as follows:

$$\xi^\alpha_0 = o^\alpha, \quad \xi^\alpha_1 = \iota^\alpha, \quad \overline{\xi}^{\dot{\alpha}}_{\dot{0}} = \overline{o}^{\dot{\alpha}}, \quad \overline{\xi}^{\dot{\alpha}}_{\dot{1}} = \overline{\iota}^{\dot{\alpha}}, \qquad (6.226)$$

$$\alpha, \beta \ldots = 0, 1, \quad \dot{\alpha}, \dot{\beta} \ldots = \dot{0}, \dot{1}.$$

From the orthogonality condition for the spinor dyad

$$\begin{aligned} \xi^0_\alpha\xi^\alpha_1 &= 1, \\ \xi^\alpha_0\xi^0_\alpha = -\xi^0_\alpha\xi^\alpha_0 &= 0, \\ \xi^\alpha_1\xi^1_\alpha &= 0. \end{aligned} \qquad (6.227)$$

$$\begin{aligned} \xi^0_\alpha\xi^\beta_0 - \xi^1_\alpha\xi^\beta_0 &= \delta^\beta_\alpha, \\ \xi^0_\alpha\xi^1_\beta - \xi^1_\alpha\xi^0_\beta &= \varepsilon_{\alpha\beta}, \end{aligned} \qquad (6.228)$$

where

$$\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = \varepsilon_{\dot\gamma\dot\delta} = \varepsilon^{\dot\gamma\dot\delta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.229}$$

we derive the normalization condition for the two-component spinors

$$o_\alpha \iota^\alpha = -\iota_\alpha o^\alpha = 1,$$
$$o^\alpha o_\alpha = -o_\alpha o^\alpha = 0, \quad \iota^\alpha \iota_\alpha = 0, \tag{6.230}$$

and also the relationships

$$\varepsilon^{\alpha\beta} = o^\alpha \iota^\beta - \iota^\alpha o^\beta, \quad \varepsilon_{\alpha\beta} = o_\alpha \iota_\beta - o_\beta \iota_\alpha, \quad \varepsilon^\beta_\alpha = o_\alpha \iota^\beta - \iota_\alpha o^\beta.$$

Spinors o^α and i^β define the components of the Newman-Penrose symbols (6.6)

$$\sigma^i_{A\dot B} = \sigma^i_{\alpha\dot\beta} \xi^\alpha_A \overline{\xi}^{\dot\beta}_{\dot B} \tag{6.231}$$

as follows:

$$\begin{aligned} \sigma^i_{0\dot 0} &= \sigma^i_{\alpha\dot\beta} o^\alpha \overline{o}^{\dot\beta} = l^i, \quad & \sigma^i_{1\dot 1} &= \sigma^i_{\alpha\dot\beta} \iota^\alpha \overline{\iota}^{\dot\beta} = n^i, \\ \sigma^i_{0\dot 1} &= \sigma^i_{\alpha\dot\beta} o^\alpha \overline{\iota}^{\dot\beta} = m^i, \quad & \sigma^i_{1\dot 0} &= \sigma^i_{\alpha\dot\beta} \iota^\alpha \overline{o}^{\dot\beta} = \overline{m}^i. \end{aligned} \tag{6.232}$$

The vectors l^i, n^i, m^i and $\overline{m}^i$ form an isotropic tetrad. The conventional tetrad $e^i{}_a$ can be made up of the vectors of an isotropic tetrad using the relationships

$$e^i_0 = (2)^{-1/2}(l^i + n^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha \overline{o}^{\dot\beta} + \iota^\alpha \overline{\iota}^{\dot\beta}),$$
$$e^i_1 = (2)^{-1/2}(m^i + \overline{m}^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha \overline{\iota}^{\dot\beta} + \iota^\alpha \overline{o}^{\dot\beta}),$$
$$e^i_2 = (2)^{-1/2} i(m^i - \overline{m}^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha \overline{\iota}^{\dot\beta} - \iota^\alpha \overline{o}^{\dot\beta}),$$
$$e^i_3 = (2)^{-1/2}(l^i - n^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha \overline{o}^{\dot\beta} - \iota^\alpha \overline{\iota}^{\dot\beta}). \tag{6.233}$$

Using the relationships

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot B\dot D}\sigma^i_{C\dot D}\nabla_k \sigma_{A\dot B i}, \tag{6.234}$$

$$\nabla_{\alpha\dot\beta} = \sigma^i_{\alpha\dot\beta}\nabla_i \tag{6.235}$$

we find the following expressions for the components of the Carmeli matrices [54]:

$$\begin{aligned} -\kappa &= o^\alpha \overline{o}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \quad & -\lambda &= \iota^\alpha \overline{o}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \\ -\rho &= \iota^\alpha \overline{o}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \quad & -\pi &= o^\alpha \overline{o}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \\ -\sigma &= o^\alpha \overline{\iota}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \quad & -\varepsilon &= o^\alpha \overline{o}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \\ -\tau &= \iota^\alpha \overline{\iota}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \quad & -\beta &= o^\alpha \overline{\iota}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} o_\gamma, \\ -\nu &= \iota^\alpha \overline{\iota}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \quad & -\gamma &= \iota^\alpha \overline{\iota}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \\ -\mu &= o^\alpha \overline{\iota}^{\dot\beta} \iota^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \quad & -\alpha &= \iota^\alpha \overline{o}^{\dot\beta} o^\gamma \nabla_{\alpha\dot\beta} \iota_\gamma, \end{aligned} \tag{6.236}$$

$$\begin{array}{ll} \Psi_0 = \Psi_{\alpha\beta\chi\delta} o^\alpha o^\beta o^\chi o^\delta, & \Psi_1 = \Psi_{\alpha\beta\chi\delta} o^\alpha o^\beta o^\chi \iota^\delta, \\ \Psi_2 = \Psi_{\alpha\beta\chi\delta} o^\alpha o^\beta \iota^\chi \iota^\delta, & \Psi_3 = \Psi_{\alpha\beta\chi\delta} o^\alpha \iota^\beta \iota^\chi \iota^\delta, \end{array} \tag{6.237}$$

$$\Psi_4 = \Psi_{\alpha\beta\chi\delta} \iota^\alpha \iota^\beta \iota^\chi \iota^\delta,$$

$$\begin{array}{ll} \Phi_{00} = \overline{\Phi}_{00} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} o^\alpha o^\beta \overline{o}^{\dot{\chi}} \overline{o}^{\dot{\delta}}, & \Phi_{01} = \overline{\Phi}_{10} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} o^\alpha o^\beta \overline{o}^{\dot{\chi}} \overline{\iota}^{\dot{\delta}}, \\ \Phi_{02} = \overline{\Phi}_{2} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} o^\alpha o^\beta \overline{\iota}^{\dot{\chi}} \overline{\iota}^{\dot{\delta}}, & \Phi_{11} = \overline{\Phi}_{11} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} o^\alpha \iota^\beta \overline{o}^{\dot{\chi}} \overline{\iota}^{\dot{\delta}}, \\ \Phi_{12} = \overline{\Phi}_{21} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} o^\alpha \iota^\beta \overline{\iota}^{\dot{\chi}} \overline{\iota}^{\dot{\delta}}, & \Phi_{22} = \overline{\Phi}_{22} = \Phi_{\alpha\beta\dot{\chi}\dot{\delta}} \iota^\alpha \iota^\beta \overline{\iota}^{\dot{\chi}} \overline{\iota}^{\dot{\delta}}. \end{array} \tag{6.238}$$

It follows from (6.236) that

$$\begin{array}{c} \nabla_{\beta\dot{\chi}} o_\alpha = \gamma o_\alpha o_\beta \overline{o}_{\dot{\chi}} - \alpha o_\alpha o_\beta \overline{\iota}_{\dot{\chi}} - \beta o_\alpha \iota_\beta \overline{o}_{\dot{\chi}} + \varepsilon o_\alpha \iota_\beta \overline{\iota}_{\dot{\chi}} - \\ - \tau \iota_\alpha o_\beta \overline{o}_{\dot{\chi}} + \rho \iota_\alpha o_\beta \overline{\iota}_{\dot{\chi}} + \sigma \iota_\alpha \iota_\beta \overline{o}_{\dot{\chi}} - \kappa \iota_\alpha \iota_\beta \overline{\iota}_{\dot{\chi}}, \end{array} \tag{6.239}$$

$$\begin{array}{c} \nabla_{\beta\dot{\chi}} \iota_\alpha = \nu o_\alpha o_\beta \overline{o}_{\dot{\chi}} - \lambda o_\alpha o_\beta \overline{\iota}_{\dot{\chi}} - \mu o_\alpha \iota_\beta \overline{o}_{\dot{\chi}} + \pi o_\alpha \iota_\beta \overline{\iota}_{\dot{\chi}} - \\ - \gamma \iota_\alpha o_\beta \overline{o}_{\dot{\chi}} + \alpha \iota_\alpha o_\beta \overline{\iota}_{\dot{\chi}} + \beta \iota_\alpha \iota_\beta \overline{o}_{\dot{\chi}} - \varepsilon \iota_\alpha \iota_\beta \overline{\iota}_{\dot{\chi}}. \end{array} \tag{6.240}$$

The components of the spinor derivative (6.114) can be represented in terms of two-component spinors as

$$\begin{array}{ll} D = -o^\alpha \overline{o}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, & \Delta = -\iota^\alpha \overline{\iota}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, \\ \delta = -o^\alpha \overline{\iota}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, & \overline{\delta} = -\iota^\alpha \overline{o}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}. \end{array} \tag{6.241}$$

In the formalism of two-component spinors there exists the so-called modified formalism [53] that takes into account the "primed" symmetry of spinor quantities. This symmetry allows the replacement

$$\begin{array}{ll} o^\alpha \to i\iota^\alpha, & \iota^\alpha \to i o^\alpha, \\ \overline{o}^{\dot{\alpha}} \to -i\overline{\iota}^{\dot{\alpha}}, & \overline{\iota}^{\dot{\alpha}} \to -i\overline{o}^{\dot{\alpha}}, \end{array} \tag{6.242}$$

where the unprimed quantities are replaced by primed ones following the rule

$$(l^i)' = n^i, \quad (m^i)' = \overline{m}^i, \quad (\overline{m}^i)' = m^i, \quad (n^i)' = l^i, \tag{6.243}$$

$$\begin{array}{lll} \nu = -\kappa', & \lambda = -\sigma', & \mu = -\rho', \\ \pi = -\tau', & \alpha = -\beta', & \gamma = -\varepsilon'. \end{array} \tag{6.244}$$

This symmetry property makes it possible to replace in (6.236) unprimed quantities by primes ones

$$\begin{array}{ll} -\kappa = \overline{o}^{\dot{\alpha}} o^\beta o^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, & \sigma' = \overline{\iota}^{\dot{\alpha}} o^\beta \iota^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, \\ -\rho = \iota^\alpha \overline{o}^{\dot{\beta}} o^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, & \tau' = o^\alpha \overline{o}^{\dot{\beta}} \iota^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, \\ -\sigma = o^\alpha \overline{\iota}^{\dot{\beta}} o^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, & -\varepsilon = o^\alpha \overline{o}^{\dot{\beta}} \iota^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, \\ -\tau = \iota^\alpha \overline{\iota}^{\dot{\beta}} o^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, & -\beta = o^\alpha \overline{\iota}^{\dot{\beta}} \iota^\gamma \nabla_{\alpha\dot{\beta}} o_\gamma, \\ \kappa' = \iota^\alpha \overline{\iota}^{\dot{\beta}} \iota^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, & \varepsilon' = \iota^\alpha \overline{\iota}^{\dot{\beta}} o^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, \\ \rho' = o^\alpha \overline{\iota}^{\dot{\beta}} \iota^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, & \beta' = \iota^\alpha \overline{o}^{\dot{\beta}} o^\gamma \nabla_{\alpha\dot{\beta}} \iota_\gamma, \end{array} \tag{6.245}$$

Here instead of 12 spinor coefficients we have only six.

The most general transformation under which spinors o^α, ι^α and the conditions (6.230) are retained is

$$o^\alpha \to C o^\alpha, \quad \iota^\alpha \to C^{-1}\iota^\alpha, \tag{6.246}$$

where C is a complex transformation that forms a subgroup of boosts and three-dimensional rotations. The components of the isotropic tetrad (6.232) vary under these transformations as follows:

$$l_i \to A^{-1} l_i, \quad n_i \to A n_i, \quad m_i \to e^{i\theta} m_i,$$
$$A = C\overline{C}, \quad e^{i\theta} = C\overline{C}^{-1}. \tag{6.247}$$

We will now define a scalar quantity with the following properties:

$$\eta \to C^p \overline{C}^q \eta. \tag{6.248}$$

This quantity is said to be a spin and boost weight scalar of the type (p, q) [53]. It follows from (6.246) that the components of the spinors o^α and ι^α are scalars of types $(1, 0)$ and $(-1, 0)$, respectively. The components of the isotropic tetrad will be

$$l^i: \ (1,1), \ n^i: \ (-1,-1), m^i: \ (1,-1), \ \overline{m}^i: \ (-1,1). \tag{6.249}$$

In respect of the transformations (6.247) all the spin coefficients (6.236) can be divided into two classes:

(a) quantities that are transformed in a uniform manner, e.g.,

$$\sigma \to (C o^\alpha)(\overline{C}^{-1}\overline{\iota}^{\dot\alpha})(C o^\beta)\nabla_{\alpha\dot\alpha}(C o_\beta) = C^3\overline{C}^{-1}\sigma; \tag{6.250}$$

(b) quantitites that are transformed in a nonuniform manner using derivatives of C, e.g.,

$$\beta \to (C o^\alpha)(\overline{C}^{-1}\overline{\iota}^{\dot\alpha})(C^{-1}\iota^\beta)\nabla_{\alpha\dot\alpha}(C o_\beta) =$$
$$= C\overline{C}^{-1}\beta + \overline{C}^{-1} o^\alpha \overline{\iota}^{\dot\alpha}\nabla_{\alpha\dot\alpha} C. \tag{6.251}$$

If we take into account "primed" symmetry and spin and boost weights, the main spinor quantities become

$$\begin{array}{llll} \kappa : (3,1), & \sigma : (3,-1), & \rho : (1,1), & \tau : (1,-1), \\ \kappa' : (-3,-1), & \sigma' : (-3,1), & \rho' : (-1,-1), & \tau' : (-1,1), \end{array}$$

$$\Psi_0 = \Psi_4' : (4,0), \quad \Psi_1 = \Psi_3' : (2,0),$$
$$\Psi_2 = \Psi_2' : (0,0), \quad \Psi_3 = \Psi_1' : (-2,0),$$
$$\Psi_4 = \Psi_0' : (-4,0), \tag{6.252}$$

$$
\begin{array}{ll}
\Phi_{00} = \overline{\Phi}_{00} = \Phi'_{22} : (2, 2), & \Phi_{01} = \overline{\Phi}_{10} = \Phi'_{21} : (2, 0), \\
\Phi_{02} = \overline{\Phi}_{20} = \Phi'_{20} : (2, -2), & \Phi_{10} = \overline{\Phi}_{01} = \Phi'_{12} : (0, 2), \\
\Phi_{11} = \overline{\Phi}_{11} = \Phi'_{11} : (0, 0), & \Phi_{12} = \overline{\Phi}_{21} = \Phi'_{10} : (0, -2), \\
\Phi_{20} = \overline{\Phi}_{02} = \Phi'_{02} : (-2, +2), & \Phi_{21} = \overline{\Phi}_{12} = \Phi'_{01} : (-2, 0),
\end{array}
$$

$$
\Phi_{22} = \overline{\Phi}_{22} = \Phi'_{00} : (-2, -2),
$$

$$
\Lambda = \overline{\Lambda} = \Lambda' = R/24 : (0, 0).
$$

For weighted quantities we will introduce new differential operators, such that their action on a scalar η of the type $\{p, q\}$ is defined as

$$
\begin{array}{ll}
P\eta = (D - p\varepsilon - q\overline{\varepsilon})\eta, & P'\eta = (\Delta + p\varepsilon' + q\overline{\varepsilon}')\eta, \\
\partial\eta = (\delta - p\beta + q\overline{\beta}')\eta, & \partial'\eta = (\overline{\delta} + p\beta - q\overline{\beta})\eta.
\end{array} \tag{6.253}
$$

Operators (6.253) have the following spin weights:

$$
\begin{array}{ll}
P : (1, 1), & \partial : (1, -1), \\
P' : (-1, -1), & \partial' : (-1, 1).
\end{array} \tag{6.254}
$$

In terms of (6.252) and differential operators (6.253), we can write the spinor equations (B^{s+}) in a simpler form [53]. Shown schematically in Fig. 6.1 are the boost and spin weights of the main spinors of the A_4 geometry.

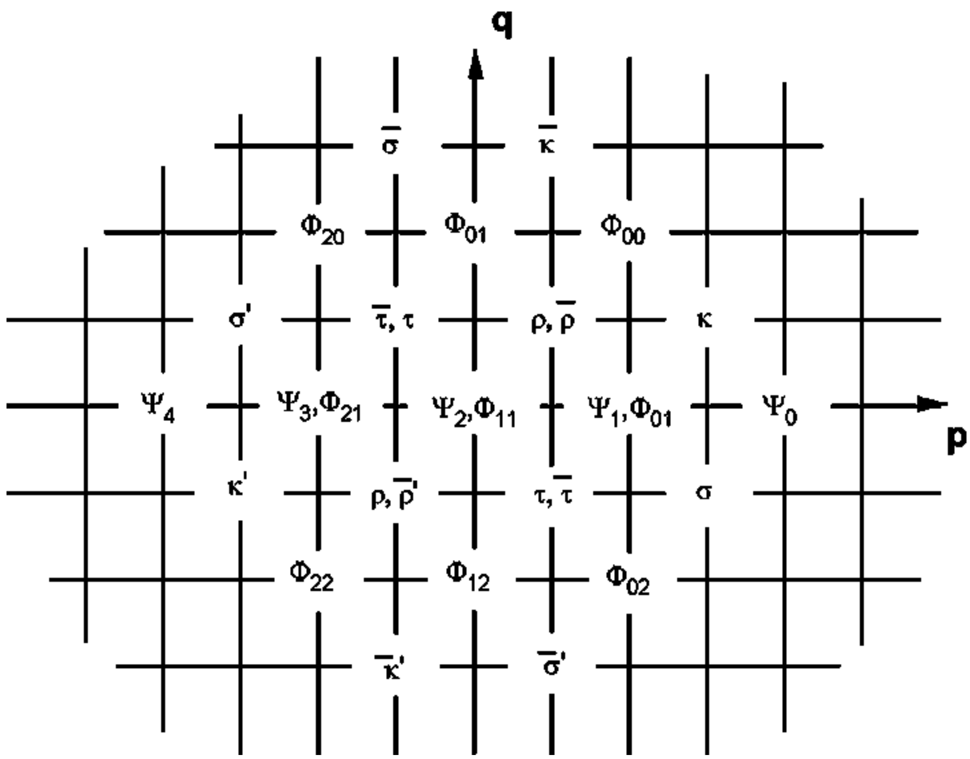

Figure 6.1: Boost and spin weights of main spinors of the A_4 geometry

Chapter 7

Construction of solutions to structural Cartan equations of the geometry of absolute parallelism

7.1 Selection of a frame of reference and specialization of Newman-Penrose symbols

The structural Cartan equations of any geometry describe the general connection between basic geometrical characteristics of a given geometry. A special solution of structural equations determines specific geometrical quantities, such as the curvature, connection, metric, etc., characteristic of a given specific solution [55]. For simplicity we will investigate the structural Cartan equations of the A_4 geometry

$$\nabla_{[k}e^a{}_{m]} - e^b{}_{[k}T^a{}_{|b|m]} = 0, \tag{A}$$

$$R^a{}_{bkm} + 2\nabla_{[k}T^a{}_{|b|m]} + 2T^a{}_{c[k}T^c{}_{|b|m]} = 0, \tag{B}$$

written in the vector basis, for their compatibility. These equations are essentially a system that in the general case includes 44 (24 equations (A) and 20 equations (B)) nonlinear partial differential equations of the first order with the following unknown functions:

(a) 6 components of anholonomic tetrad

$$e^i{}_a = \nabla_a x^i; \tag{7.1}$$

(b) 24 components of the Ricci rotation coefficients

$$T^a{}_{bk} = e^j{}_b \nabla_k e^a{}_j; \tag{7.2}$$

(c) 20 components of the Riemann tensor

$$R^a{}_{bkm}\,. \tag{7.3}$$

Thus, in the general case we have 44 equations for 50 unknown functions. This gives us some freedom in choosing a reference frame x^i, of the tetrad $e^i{}_a$, and also of the quantities $T^a{}_{bk}$ and $R^a{}_{bkm}$. Therefore, search for specific solution to the set of equations (A) and (B) should rather be referred to as "constructing solutions."

When constructing solutions it is convenient to represent the structural Cartan equations of the geometry of absolute parallelism in the spinor Δ-basis in terms of Carmeli matrices

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} -$$

$$-(T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}\,, \tag{A^s}$$

$$R_{F\dot{E}D\dot{B}} = \partial_{D\dot{B}}T_{F\dot{E}} - \partial_{F\dot{E}}T_{D\dot{B}} - (T_{D\dot{B}})_F{}^S T_{S\dot{E}} - (T^+_{\dot{B}D})^{\dot{F}}{}_{\dot{E}}T_{F\dot{F}} +$$

$$+(T_{F\dot{E}})_D{}^S T_{S\dot{B}} + (T^+_{\dot{E}F})^{\dot{F}}{}_{\dot{B}}T_{D\dot{F}} + [T_{F\dot{E}}, T_{D\dot{B}}]. \tag{B^s+}$$

where the components of the traceless 2×2 matrices $R_{F\dot{E}D\dot{B}}$ and $T_{F\dot{E}}$ are found from the relationships (6.88) and (6.103). Let us now find the Newman-Penrose symbols via the spinor representation of the invariant Haiashi derivative

$$\sigma_{A\dot{B}}{}^i = \nabla_{A\dot{B}}x^i = \partial_{A\dot{B}}x^i, \tag{7.4}$$

where the components of the spinor derivative $\partial_{A\dot{B}}$ are denoted as

$$\partial_{A\dot{B}} = \begin{array}{c|cc} & \multicolumn{2}{c}{\dot{B}} \\ A & \dot{0} & \dot{1} \\ \hline 0 & D & \delta \\ 1 & \bar{\delta} & \Delta \end{array} \tag{7.5}$$

From the relationships (7.4)-(7.5) and

$$\sigma^i{}_{A\dot{B}} = \begin{array}{c|cc} & \multicolumn{2}{c}{\dot{B}} \\ \hline A & \dot{0} & \dot{1} \\ \hline 0 & l^i = (Y^0, V, Y^2, Y^3) & m^i = (\xi^0, \omega, \xi^2, \xi^3) \\ 1 & \bar{m}^i = (\bar{\xi}^0, \bar{\omega}, \bar{\xi}^2, \bar{\xi}^3) & n^i = (X^0, U, X^2, X^3) \end{array} \tag{7.6}$$

we obtain

$$l^i = Dx^i, \quad n^i = \Delta x^i, \quad m^i = \delta x^i, \quad \bar{m}^{i-} = \delta x^i, \tag{7.7}$$

and also

$$\begin{array}{c} Y^0 = Dx^0, X^0 = \Delta x^0, \xi^0 = \delta x^0, \bar{\xi}^0 = \bar{\delta} x^0, \\ V = Dx^1, U = \Delta x^1, \omega = \delta x^1, \bar{\omega} = \bar{\delta} x^1, \\ Y^2 = Dx^2, X^2 = \Delta x^2, \xi^2 = \delta x^2, \bar{\xi}^2 = \bar{\delta} x^2, \\ Y^3 = Dx^3, X^3 = \Delta x^3, \xi^3 = \delta x^3, \bar{\xi}^3 = \bar{\delta} x^3. \end{array} \tag{7.8}$$

From the equality
$$\partial_{A\dot{B}} = \sigma_{A\dot{B}}{}^i \nabla_i = \sigma_{A\dot{B}}{}^i{}_{,i} \qquad (7.9)$$
and the relationships (7.5) and (7.6) it follows
$$D = l^i \nabla_i, \quad \Delta = n^i \nabla_i, \quad \delta = m^i \nabla_i, \quad \overline{\delta} = \overline{m}^i \nabla_i \qquad (7.10)$$
or
$$\begin{aligned} D &= V \frac{\partial}{\partial x^1} + Y^\alpha \frac{\partial}{\partial x^\alpha}, \\ \Delta &= V \frac{\partial}{\partial x^1} + X^\alpha \frac{\partial}{\partial x^\alpha}, \\ \delta &= w \frac{\partial}{\partial x^1} + \xi^\alpha \frac{\partial}{\partial x^\alpha}, \\ \overline{\delta} &= \overline{w} \frac{\partial}{\partial x^1} + \overline{\xi}^\alpha \frac{\partial}{\partial x^\alpha}, \\ &\alpha = 0, 2, 3. \end{aligned} \qquad (7.11)$$

Using these relationships, we write the vectors that make up the matrix (7.6) as
$$\begin{aligned} l^i &= V \delta^i_1 + Y^\alpha \delta^i_\alpha, \\ n^i &= U \delta^i_1 + X^\alpha \delta^i_\alpha, \\ m^i &= \omega \delta^i_1 + \xi^\alpha \delta^i_\alpha, \\ \overline{m}^i &= \overline{\omega} \delta^i_1 + \overline{\xi}^\alpha \delta^i_\alpha. \end{aligned} \qquad (7.12)$$

From the orthogonality condition for the Newman-Penrose symbols
$$\sigma_i^{A\dot{B}} \sigma^j_{A\dot{B}} = \delta^j_i, \qquad (7.13)$$
$$\sigma_i^{A\dot{B}} \sigma^i_{C\dot{E}} = \delta^A{}_C \delta^{\dot{B}}{}_{\dot{E}}. \qquad (7.14)$$
follow the orthogonality conditions for the vectors (7.12)
$$\begin{aligned} l_i l^i = m_i m^i = \overline{m}_i \overline{m}^i = n_i n^i = 0, \\ l_i n^i = -m_i \overline{m}^i = 1, \\ l_i m^i = l_i \overline{m}^i = n_i m^i = n_i \overline{m}^i = 0. \end{aligned} \qquad (7.15)$$
And from the formulas
$$g_{ij} = \varepsilon_{AC} \varepsilon_{\dot{B}\dot{D}} \sigma_i^{A\dot{B}} \sigma_j^{C\dot{D}}, \qquad (7.16)$$
$$\varepsilon_{00} = \varepsilon_{11} = 0, \quad \varepsilon_{01} = -1, \quad \varepsilon_{10} = 1,$$
we find
$$g^{ij} = l^i n^j + l^j n^i - m^i \overline{m}^j - m^j \overline{m}^i. \qquad (7.17)$$

Vectors that meet the orthogonality conditions (7.15) are null vectors and in physics they are normally associated with propagation of radiation (i.e., with matter that has no rest mass), where concepts of wave fronts, waves, rays, etc, hold. In the process a family of null hypersurfaces $u(x^i) = \text{const}$ is introduced.

We will take the vector l_i to be orthogonal to these hypersurfaces

$$l_i = u_{,i}. \tag{7.18}$$

Further, we will select the coordinates so that [60]

$$\begin{array}{ll} x^0 = u, & \\ x^1 = r, & \text{where } r \text{ is the affine parameter along the null geodesics} \\ x^2, & \\ x^3, & \text{where } x^{2,3} \text{ assign numbers to rays on each hypersurface and are constant along the rays.} \end{array} \tag{7.19}$$

When selecting the coordinates, the vector l^i and l_i look like:

$$l_i = u_{,i} = x^0_{,i} = \delta^0_i, \tag{7.20}$$

$$l^i = \frac{dx^i}{dx^1} = \frac{dx^i}{dr} = \delta_1{}^i \tag{7.21}$$

or

$$l^i = (0, 1, 0, 0), \quad l_i = (1, 0, 0, 0). \tag{7.22}$$

From the orthogonality conditions

$$l_i n^i = 1, \quad l_i m^i = 0$$

it follows that

$$\begin{aligned} n^i &= (1, U, X^2, X^3), \\ m^i &= (0, \omega, \xi^2, \xi^3), \end{aligned} \tag{7.23}$$

and the relationships (7.11) become

$$\begin{aligned} D &= V\frac{\partial}{\partial x^1} = \frac{\partial}{\partial r}, \\ \Delta = U\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^0} + X^\beta\frac{\partial}{\partial x^\beta} &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^\beta\frac{\partial}{\partial x^\beta}, \\ \delta = \omega\frac{\partial}{\partial x^1} + \xi^\beta\frac{\partial}{\partial x^\beta} &= \omega\frac{\partial}{\partial r} + \xi^\beta\frac{\partial}{\partial x^\beta}, \\ \delta = \overline{\omega}\frac{\partial}{\partial x^1} + \overline{\xi}^\beta\frac{\partial}{\partial x^\beta} &= \overline{\omega}\frac{\partial}{\partial r} + \overline{\xi}^\beta\frac{\partial}{\partial x^\beta}, \\ \beta &= 2, 3. \end{aligned} \tag{7.24}$$

Moreover, the vectors (7.12) will be given by

$$\begin{aligned} l^i &= \delta_1{}^i, \\ n^i &= \delta_0{}^i + U\delta^i_1 + X^\beta\delta^i_\beta, \\ m^i &= \omega\delta_1{}^i + \xi^\beta\delta^i_\beta, \\ \overline{m}^i &= \overline{\omega}\delta^i_1 + \overline{\xi}^\beta\delta^i_\beta, \end{aligned} \tag{7.25}$$

$$\beta = 2,3.$$

Since

$$l^i = g^{ik} l_k = g^{ik} \delta_k^0 = g^{io} = \delta_1^i,$$

the metric tensor has the following structure [56]:

$$g^{ik} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{11} & g^{12} & g^{13} \\ 0 & g^{12} & g^{22} & g^{23} \\ 0 & g^{13} & g^{23} & g^{33} \end{pmatrix}. \tag{7.26}$$

Using the relationship (7.17) and (7.25), we get

$$\begin{aligned} g^{22} &= 2(U - \overline{\omega}\omega), && (a) \\ g^{2\beta} &= X^\beta - (\xi^\beta \overline{\omega} + \overline{\xi}^\beta \omega), && (b) \\ g^{\gamma\delta} &= -(\xi^\gamma \overline{\xi}^\delta + \overline{\xi}^\gamma \xi^\delta), && (c) \\ & && \gamma, \delta, \beta = 2,3. \end{aligned} \tag{7.27}$$

As is seen from the above reasoning, the coordinates (7.19) selected and the specialization of the Newman-Penrose symbols using the relationship (7.18) made it possible to us to derive the dependence (7.27) and the general form (7.26) of the metric tensor g^{ik} of the A_4 geometry.

7.2 Specialization of the spinor components of the Ricci rotation coefficients

The spinor structural Cartan equations (A^s) and (B^s) of the geometry A_4 can be viewed as a matrix of possible geometries of absolute parallelism that differ in specific set of spinor geometrical characteristics. Therefore, we will assume the solution of the spinor structural equations (A^s) and (B^s) (in this reference frame) to be a set of variables consisting in the general case of:

(a) 6 independent components of the Newman-Penrose symbols

$$\sigma^i_{A\dot{B}}; \tag{7.28}$$

(b) 24 independent spinor components of the Ricci rotation coefficients

$$T_{A\dot{B}}, \quad T^+_{\dot{B}A}; \tag{7.29}$$

(c) 20 independent spinor components of the independent spinor components of the Riemannian tensor

$$R_{A\dot{B}C\dot{D}}, \quad R^+_{\dot{B}A\dot{D}C}, \tag{7.30}$$

that transform the equations (A^s) and (B^s) into identities when substituted into these equations.

In our search for solutions to the structural equations ($\boldsymbol{A}^s$) and ($\boldsymbol{B}^s$) we will rely on the symmetry conditions, and also on physical arguments, e.g., we will subject the Riemannian tensor to the conditions of Einstein's vacuum

$$R_{ij} = 0, \tag{7.31}$$

which can be represented in terms of Carmeli matrices (7.103), (7.214) and (7.217) as

$$R_{A\dot{B}C\dot{D}} = C_{A\dot{B}C\dot{D}} = 0.$$

We will now consider the limitations that can be imposed on the components of the matrices (6.88), using physical reasoning. To this end, we will turn to the relationship

$$\nabla_k \sigma^i{}_{C\dot{D}} = \left((T_{A\dot{B}})^P_C \sigma^i{}_{P\dot{D}} + \sigma^i{}_{C\dot{R}} (T^+_{\dot{B}A})^{\dot{R}}_{\dot{D}} \right) \sigma^{A\dot{B}}_k \tag{7.32}$$

or

$$(T_{A\dot{B}})^P_C \sigma^i{}_{P\dot{D}} + \sigma^i{}_{C\dot{R}} (T^+_{\dot{B}A})^{\dot{R}}_{\dot{D}} = \nabla_k \sigma^i{}_{C\dot{D}} \sigma^k{}_{A\dot{B}}. \tag{7.33}$$

From (7.6), (6.88), and (7.33) we get

$$(7.34)$$

$$l^k\nabla_k l^i = (\varepsilon+\overline{\varepsilon})l^i - \overline{\kappa}m_i - \kappa\overline{m}^i, \tag{7.34a}$$

$$n^k\nabla_k l^i = (\gamma+\overline{\gamma})l^i - \overline{\tau}m_i - \tau\overline{m}^i, \tag{7.34b}$$

$$m^k\nabla_k l^i = (\overline{\alpha}+\beta)l^i - \overline{\rho}m_i - \sigma\overline{m}^i, \tag{7.34c}$$

$$\overline{m}^k\nabla_k l^i = (\alpha+\overline{\beta})l^i - \rho m_i - \overline{\sigma}m^i, \tag{7.34d}$$

$$(7.35)$$

$$l^k\nabla_k n^i = -(\varepsilon+\overline{\varepsilon})n^i + \pi m_i + \overline{\pi m}^i, \tag{7.35a}$$

$$n^k\nabla_k n^i = -(\gamma+\overline{\gamma})n^i + \nu m_i + \overline{\nu m}^i, \tag{7.35b}$$

$$m^k\nabla_k n^i = -(\overline{\alpha}+\beta)n^i + \mu m_i + \overline{\lambda}\overline{m}^i, \tag{7.35c}$$

$$\overline{m}^k\nabla_k n^i = -(\overline{\alpha}+\overline{\beta})n^i + \overline{\mu m}_i + \lambda m^i, \tag{7.35d}$$

$$(7.36)$$

$$l^k\nabla_k m^i = (\varepsilon-\overline{\varepsilon})m^i + \overline{\pi}l_i - \kappa n^i, \tag{7.36a}$$

$$n^k\nabla_k m^i = (\gamma-\overline{\gamma})m^i + \overline{\nu}l_i - \tau n^i, \tag{7.36b}$$

$$m^k\nabla_k m^i = (\beta-\overline{\alpha})m^i + \overline{\lambda}l_i - \sigma n^i, \tag{7.36c}$$

$$\overline{m}^k\nabla_k m^i = (\alpha-\overline{\beta})m^i + \overline{\mu}l_i - \rho n^i, \tag{7.36d}$$

$$(7.37)$$

$$l^k\nabla_k \overline{m}^i = (\overline{\varepsilon}-\varepsilon)\overline{m}^i + \pi l_i - \overline{\kappa}n^i, \tag{7.37a}$$

$$n^k\nabla_k \overline{m}^i = (\overline{\gamma}-\gamma)\overline{m}^i + \nu l_i - \overline{\tau}n^i, \tag{7.37b}$$

$$m^k\nabla_k \overline{m}^i = (\overline{\alpha}-\beta)\overline{m}^i + \mu l_i - \overline{\rho}n^i, \tag{7.37c}$$

$$\overline{m}^k\nabla_k \overline{m}^i = (\overline{\beta}-\alpha)\overline{m}^i + \lambda l_i - \overline{\sigma}n^i. \tag{7.37d}$$

Further, using the orthogonality condition (7.15), we have

$$\begin{array}{ccc} \kappa = \nabla_k l_i m^i l^k, & \nu = \nabla_k n_i \overline{m}^i n^k, & \rho = \nabla_k l_i m^i \overline{m}^k, \\ \mu = -\nabla_k n_i m^i m^k, & \sigma = \nabla_k l_i m^i m^k, & \lambda = -\nabla_k n_i \overline{m}^i \overline{m}^k, \\ \tau = \nabla_k l_i m^i n^k, & \pi = -\nabla_k n_i \overline{m}^i l^k, & \end{array}$$

$$(7.38)$$

$$\alpha = \frac{1}{2}(\nabla_k l_i n^i \overline{m}^k - \nabla_k m_i \overline{m}^i \overline{m}^k),$$

$$\beta = \frac{1}{2}(\nabla_k l_i n^i m^k - \nabla_k m_i \overline{m}^i m^k),$$

$$\gamma = \frac{1}{2}(\nabla_k l_i n^i n^k - \nabla_k m_i \overline{m}^i n^k),$$

$$\varepsilon = \frac{1}{2}(\nabla_k l_i n^i n^k - \nabla_k m_i \overline{m}^i l^k).$$

On the other hand, we can write (7.32) as

$$(7.39)$$

$$\begin{aligned}
\nabla_k l_j = \; & (\gamma+\overline{\gamma}) l_k l_j - \\
& -\overline{\tau} l_k m_j - \tau l_k \overline{m}_j + (\varepsilon+\overline{\varepsilon}) n_k l_j - \overline{\kappa} n_k m_j - \\
& -\kappa n_k \overline{m}_j - (\alpha+\overline{\beta}) m_k l_j + \overline{\sigma} m_k m_j + \\
& +\rho m_k \overline{m}_j - (\overline{\alpha}+\beta)\overline{m}_k l_j + \\
& +\overline{\rho}\overline{m}_k m_j + \sigma \overline{m}_k \overline{m}_j,
\end{aligned} \tag{7.39a}$$

$$\begin{aligned}
\nabla_k n_j = \; & -(\gamma+\overline{\gamma}) l_k n_j + \nu l_k m_j + \\
& +\overline{\nu} l_k \overline{m}_j - (\varepsilon+\overline{\varepsilon}) n_k n_j + \pi n_k m_j + \\
& +\overline{\pi} n_k \overline{m}_j + (\alpha+\overline{\beta}) m_k n_j - \\
& -\lambda m_k m_j - \overline{\mu} m_k \overline{m}_j + (\overline{\alpha}+\beta)\overline{m}_k n_j - \\
& -\mu \overline{m}_k m_j - \overline{\lambda}\overline{m}_k \overline{m}_j,
\end{aligned} \tag{7.39b}$$

$$\begin{aligned}
\nabla_k m_j = \; & (\gamma-\overline{\gamma}) l_k m_j + \overline{\nu} l_k l_j - \\
& -\tau l_k n_j + (\varepsilon-\overline{\varepsilon}) n_k m_j + \overline{\pi} n_k l_j - \\
& -\kappa n_k n_j + (\overline{\alpha}-\beta)\overline{m}_k m_j - \\
& -\overline{\mu} m_k l_j + \rho m_k n_j + (\overline{\beta}-\alpha) m_k m_j - \\
& -\overline{\mu}\overline{m}_k l_j + \sigma \overline{m}_k n_j,
\end{aligned} \tag{7.39c}$$

$$\begin{aligned}
\nabla_k \overline{m}_j = \; & (\overline{\gamma}-\gamma) l_k \overline{m}_j + \nu l_k l_j - \overline{\tau} l_k n_j + \\
& +(\overline{\varepsilon}-\varepsilon) n_k \overline{m}_j - \pi n_k l_j - \\
& -\overline{\kappa} n_k n_j + (\alpha-\overline{\beta}) m_k \overline{m}_j - \\
& -\mu \overline{m}_k l_j + \overline{\rho}\overline{m}_k n_j + (\beta-\overline{\alpha})\overline{m}_k \overline{m}_j - \\
& -\lambda m_k l_j + \overline{\sigma} m_k n_j,
\end{aligned} \tag{7.39d}$$

Alternating these relationships in the indices k and j gives

$$(7.40)$$

$$\nabla_{[k} l_{j]} = -2\Re(\varepsilon) l_{[k} n_{j]} - (\overline{\tau}-\alpha-\overline{\beta}) l_{[k} m_{j]} - \tag{7.40a}$$

$$-(\tau-\overline{\alpha}-\beta) l_{[k}\overline{m}_{j]} - \overline{\kappa} n_{[k} m_{j]} - \kappa n_{[k}\overline{m}_{j]} + 2i\Im(\rho) m_{[k}\overline{m}_{j]},$$

$$\nabla_{[k} n_{j]} = -2\Re(\gamma) l_{[k} n_{j]} - (\pi-\alpha-\overline{\beta}) n_{[k} m_{j]} - \tag{7.40b}$$

$$-(\overline{\pi}-\overline{\alpha}-\beta) n_{[k}\overline{m}_{j]} + \nu l_{[k} m_{j]} - \overline{\nu} l_{[k}\overline{m}_{j]} + 2i\Im(\mu) m_{[k}\overline{m}_{j]},$$

$$\nabla_{[k} m_{j]} = -(\overline{\pi}+\tau) l_{[k} n_{j]} + \left(2i\Im(\gamma)+\overline{\mu}\right) l_{[k} m_{j]} + \tag{7.40c}$$

$$+\overline{\lambda} l_{[k}\overline{m}_{j]} + \left(2i\Im(\varepsilon)-\rho\right) n_{[k} m_{j]} - \sigma n_{[k}\overline{m}_{j]} - (\overline{\alpha}-\beta) m_{[k}\overline{m}_{j]},$$

$$\nabla_{[k}\overline{m}_{j]} = -(\pi+\overline{\tau}) l_{[k} n_{j]} + \left(-2i\Im(\gamma)+\mu\right) l_{[k}\overline{m}_{j]} + \tag{7.40d}$$

$$+\lambda l_{[k}\overline{m}_{j]} + \left(-2i\Im(\varepsilon)-\overline{\rho}\right) n_{[k}\overline{m}_{j]} - \overline{\sigma} n_{[k} m_{j]} - (\alpha-\overline{\beta})\overline{m}_{[k} m_{j]}.$$

Convoluting the equations (7.39), we arrive at

$$(7.41)$$

$$\nabla_k l^k = -(\rho + \overline{\rho}) + \varepsilon + \overline{\varepsilon}, \qquad (7.41a)$$

$$\nabla_k n^k = -(\gamma + \overline{\gamma}) + \mu + \overline{\mu}, \qquad (7.41b)$$

$$\nabla_k m^k = -\overline{\alpha} + \overline{\pi} - \tau + \beta, \qquad (7.41c)$$

$$\nabla_k \overline{m}^k = -\alpha + \pi + \overline{\tau} + \overline{\beta}. \qquad (7.41d)$$

Relationships (7.34)-(7.41) appear to be quite useful for the specialization of the spinor components of the Ricci rotation coefficients. Really, we will require, for instance, that the isotropic vector l^i should obey the equations of the geodesics of Einstein's gravitation theory

$$k^k \nabla_k l^i = 0. \qquad (7.42)$$

It follows then from (7.34a) that

$$(\varepsilon + \overline{\varepsilon}) l^i - \overline{\kappa} m_i - \kappa \overline{m}^i = 0. \qquad (7.43)$$

Clearly, the relationship (7.43) holds, if the spinor components of the Ricci tensor have the following limitations:

$$(\varepsilon + \overline{\varepsilon}) = 0, \quad \kappa = \overline{\kappa} = 0. \qquad (7.44)$$

The conditions of parallel transfer of the vectors m^i, $\overline{m}^i$ and n^i along l^k in Einstein's gravitation theory become

$$l^k \nabla_k m^i = 0, \quad l^k \nabla_k \overline{m}^i = 0, \quad l^k \nabla_k n^i = 0.$$

It follows from (7.35a), (7.36a) and (7.37a) that these relationships are valid if

$$\kappa = \overline{\kappa} = \pi = \overline{\pi} = \varepsilon = \overline{\varepsilon} = 0. \qquad (7.45)$$

The isotropic vector l^i is connected with the three optical parameters [40]:
(a) extension

$$\theta = (\rho + \overline{\rho})\frac{1}{2} = 0,5 \nabla_i l^i; \qquad (7.46)$$

(b) rotation

$$\omega = (\rho - \overline{\rho})(2)^{-1/2} = \left(\frac{1}{2}\nabla_{[k} l_{i]} \nabla^k l^i\right)^{1/2}; \qquad (7.47)$$

(c) shift

$$|\hat{\sigma}| = (|\sigma\overline{\sigma}|)^{1/2} = \left(\frac{1}{2}\nabla_{(k} l_{i)} \nabla^k l^i - \theta^2\right)^{1/2}. \qquad (7.48)$$

Since we have taken l^i to be a gradient vector ($l_i = u_{,i}$), it follows from (7.47) that

$$\omega = 0, \quad \rho = \overline{\rho}. \tag{7.49}$$

Furthermore, in that case we have [12]

$$\tau = \overline{\alpha} + \beta. \tag{7.50}$$

7.3 Specialization of the spinor components of the Riemann tensor

It would be a good idea in our search for physically meaningful specific solutions to the structural Cartan equations of the A_4 geometry to use completely geometrized Einstein's equations

$$2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}. \tag{7.51}$$

It was shown in Chapter 6 that

$$\sigma^{A\dot{C}}{}_j\sigma^{B\dot{D}}{}_m(2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}) = R_{jm} - \frac{1}{2}g_{jm}R, \tag{7.52}$$

$$\sigma^{A\dot{C}}{}_j\sigma^{B\dot{D}}{}_m\nu T_{A\dot{C}B\dot{D}} = \nu T_{jm}, \tag{7.53}$$

where the geometrized matter energy-momentum tensor T_{jm} is derived from (7.221). Looking at various types of geometrized tensors (7.221), such as, e.g;

(a) energy-momentum tensor of the homogeneous A_4 space

$$T^{(1)}_{jm} = -\tilde{\Lambda}g_{jm}, \quad \tilde{\Lambda} = const; \tag{7.54}$$

(b) Einstein's vacuum tensor

$$T^{(2)}_{jm} = 0; \tag{7.55}$$

(c) energy-momentum tensor of isotropic radiation

$$T^{(3)}_{jm} = \rho l_j l_m, \quad l^i l_i = 0 \tag{7.56}$$

and so on, we will obtain various limitations to the spinor components of the matrix $R_{A\dot{B}C\dot{D}}$.

From the relationships (7.51)-(7.53) for tensors of the form (7.54) we will find the following limitations on the components of the matrices (6.103)

$$\Phi_{00} = \Phi_{22} = \Phi_{02} = \Phi_{20} = \Phi_{11} = \Phi_{01} = \Phi_{10} = \Phi_{12} = \Phi_{21} = 0,$$

$$\Psi_0 \neq 0, \quad \Psi_1 \neq 0, \quad \Psi_2 \neq 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0, \tag{7.57}$$

$$\tilde{\Lambda} = \frac{R}{4} = 6\Lambda. \qquad (7.58)$$

At the same time, the condition (7.54) imposes, via (6.221), limitations on the components of the matrices (6.88).

In the case of Einstein's vacuum the conditions (7.55) should be regarded as equations to be satisfied by the components of the matrices (6.88). In the process, in addition to (7.57), we will get

$$\Lambda = 0. \qquad (7.59)$$

For tensors of the form (7.56) we have

$$\begin{gathered}\Phi_{00} = \Phi_{02} = \Phi_{20} = \Phi_{11} = \Phi_{01} = \Phi_{10} = \Phi_{12} = \Phi_{21} = \Lambda = 0, \\ \Phi = \Phi_{22} = \frac{\nu\rho}{2}, \\ \Psi_0 \neq 0, \quad \Psi_1 \neq 0, \quad \Psi_2 \neq 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0.\end{gathered} \qquad (7.60)$$

To get an insight into the physical meaning of each spinor component of the Weyl tensor Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4, we will consider five cases:

(a) $\Psi_0 \neq 0$, the other components are zero;
(b) $\Psi_1 \neq 0$, same as above;
(c) $\Psi_2 \neq 0$, - " -;
(d) $\Psi_3 \neq 0$, - " -;
(e) $\Psi_4 \neq 0$, - " -.

In each of the five cases the components of the Weyl tensor have the following algebraic properties according to Petrov:

(a) N type (or {4}) [57, 58] with the propagation vector n_i;
(b) III type (or {31}) with the propagation vector n_i;
(c) D type (or {22}) with the propagation vector l_i and n_i;
(d) III type (or {31}) with the propagation vector l_i;
(e) N type (or {4}) with the propagation vector l_i.

The propagation vector is meant to be the main light direction [40]. If in the A_4 space the condition of Einstein's vacuum $R_{jm} = 0$ is met, and the vector l_i meets the equations

$$l_{[i}R_{j]km[n}l_{s]}l^k l^m = 0, \qquad (7.61)$$

then the vector l_i corresponds to one of the four main light directions of the Riemannian tensor, and we have

$$\Psi_0 = 0. \qquad (7.62)$$

If two or more of the main light directions point along the propagation vector l_i, then

$$R_{ijk[m}l_{n]}l^j l^k = 0 \qquad (7.63)$$

or

$$\Psi_0 = \Psi_1 = 0. \qquad (7.64)$$

According to the Goldberg-Sax theorem [59], it follows from (7.64) that

$$\sigma = \kappa = 0. \tag{7.65}$$

A simple proof of the theorem is given in [40]. It relies on the second Bianchi identities (D^{s+}).

Similarly, it can be shown that from the condition

$$\Psi_3 = \Psi_4 = 0 \tag{7.66}$$

we get

$$\nu = \lambda = 0. \tag{7.67}$$

7.4 Construction of the asymptotic behavior of insular-type geometries

A_4 geometry is said to be an insular-type geometry, if at infinity its main characteristics (metric, connection, curvature) are identical to those of a flat space.

We will also assume that the conditions of Einstein's vacuum (7.31) and relationships (7.45), (7.49), (7.50) are valid. With these assumptions the structural Cartan equations of the A_4 geometry (A^{s+}), (B^{s+}) and the second Bianchi identities (D^{s+}) can conveniently be split into the following three groups of equations:

7.4.1 Radial equations containing derivatives with respect to r

$$\left.\begin{array}{ll} D\xi^\alpha = \rho\xi^\alpha + \sigma\overline{\xi}^\alpha, & (7.68a)\\ D\omega = \rho\omega + \sigma\overline{\omega} - (\overline{\alpha} + \beta), & (7.68b)\\ DX^\alpha = (\overline{\alpha} + \beta)\overline{\xi}^\alpha + (\alpha + \overline{\beta})\overline{\xi}^\alpha, & (7.68c)\\ DU = (\overline{\alpha} + \beta)\overline{\omega} + (\alpha + \overline{\beta})\omega - (\gamma + \overline{\gamma}), & (7.68d) \end{array}\right| \tag{7.68}$$

$$\left.\begin{array}{ll} D\rho = \rho^2 + \sigma\overline{\sigma}, & (7.69a)\\ D\sigma = 2\rho\sigma + \Psi_0, & (7.69b)\\ D\tau = \tau\rho + \overline{\tau}\sigma + \Psi_1, & (7.69c)\\ D\alpha = \alpha\rho + \beta\overline{\sigma}, & (7.69d)\\ D\beta = \beta\rho + \alpha\sigma + \Psi_1, & (7.69e)\\ D\gamma = \tau\alpha + \overline{\tau}\beta + \Psi_2, & (7.69f)\\ D\lambda = \lambda\rho + \mu\overline{\sigma}, & (7.69g)\\ D\mu = \mu\rho + \lambda\sigma + \Psi_2, & (7.69h)\\ D\nu = \tau\lambda + \overline{\tau}\mu + \Psi_3, & (7.69i) \end{array}\right| \tag{7.69}$$

$$
\left.\begin{array}{lr}
D\Psi_1 - \overline{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0, & (7.70a)\\
D\Psi_2 - \overline{\delta}\Psi = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0, & (7.70b)\\
D\Psi_3 - \overline{\delta}\Psi_2 = 2\rho\Psi_3 - 2\alpha\Psi_1, & (7.70c)\\
D\Psi_4 - \overline{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2. & (7.70d)
\end{array}\right| \quad (7.70)
$$

7.4.2 Nonradial equations

$$
\left.\begin{array}{lr}
\delta X^\alpha - \Delta\xi^\alpha = (\mu + \overline{\gamma} - \gamma)\overline{\xi^\alpha} + \lambda\xi^\alpha, & (7.71a)\\
\delta\overline{\xi}^\alpha - \overline{\delta}\xi^\alpha = (\overline{\beta} - \alpha)\xi^\alpha + (\overline{\alpha} - \beta)\overline{\xi}^\alpha, & (7.71b)\\
\delta\overline{\omega} - \overline{\delta}\omega = (\overline{\beta} - \alpha)\omega + (\overline{\alpha} - \beta)\overline{\omega} + (\mu - \overline{\mu}), & (7.71c)\\
\delta U - \Delta\omega = (\mu + \overline{\gamma} - \gamma)\omega + \overline{\lambda}\overline{\omega} - \nu, & (7.71d)
\end{array}\right| \quad (7.71)
$$

$$
\left.\begin{array}{lr}
\Delta\lambda - \overline{\delta}\nu = 2\alpha\nu + (\overline{\gamma} - 3\gamma - \mu - \overline{\mu})\lambda - \Psi_4, & (7.72a)\\
\delta\rho - \delta\sigma = (\beta + \overline{\alpha})\rho + (\overline{\beta} - 3\alpha)\sigma - \Psi_1, & (7.72b)\\
\delta\alpha - \overline{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - \Psi_2, & (7.72c)\\
\delta\lambda - \overline{\delta}\mu = (\alpha + \overline{\beta})\mu + (\overline{\alpha} - 3\beta)\lambda - \Psi_3, & (7.72d)\\
\delta\nu - \Delta\mu = \mu\gamma + \overline{\gamma}\mu + \mu^2 - 2\nu\beta + \lambda\overline{\lambda}, & (7.72f)\\
\delta\gamma - \Delta\beta = \tau\mu + (\mu - \gamma + \overline{\gamma})\beta - \sigma\nu + \overline{\lambda}\alpha, & (7.72g)\\
\delta\tau - \Delta\sigma = 2\tau\beta + (\overline{\gamma} + \mu - 3\gamma)\sigma + \overline{\lambda}\rho, & (7.72h)\\
\Delta\rho - \overline{\delta}\tau = (\gamma + \overline{\gamma} - \overline{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2, & (7.72i)\\
\Delta\alpha - \overline{\delta}\gamma = (\overline{\gamma} - \gamma - \overline{\mu})\alpha + \rho\nu - \tau\lambda - \lambda\beta - \Psi_3. & (7.72j)
\end{array}\right| \quad (7.72)
$$

7.4.3 U-derivative equations

$$
\left.\begin{array}{lr}
\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2, & (7.73a)\\
\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + (2\gamma - 2\mu)\Psi_1 + 2\sigma\Psi_3 - 3\tau\Psi_2, & (7.73b)\\
\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + (-2\tau + 2\beta)\Psi_3 + \sigma\Psi_4, & (7.73c)\\
\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - (2\gamma + 4\mu)\Psi_3 + (-\tau + 4\beta)\Psi_4. & (7.73d)
\end{array}\right| \quad (7.73)
$$

Suppose now that the structural Cartan equations of the A_4 geometry describe an insular radiating system. In the process, the quantity Ψ_0 behaves at an asymptotic along the coordinate r as

$$\Psi_0 = o(r^{-5}), \qquad (7.74)$$

whereas

$$D\Psi_0 = o(r^{-6}). \qquad (7.75)$$

The conditions (7.74) has been chosen on purely physical grounds in such a manner that the quadrupole radiation in a linear approximation of Einstein's gravitational theory would correspond to the asymptotic. It is clear that we could use another kind of asymptotic and have other asymptotic properties for the insular A_4 geometry. It is precisely what it mean by "constructing a geometry."

Let now uniform perturbations in the coordinates x^2 and x^3 do not change the nature of the asymptotic (7.74) and (7.75), i.e.,

$$\begin{aligned} d_\alpha \Psi_0 &= o(r^{-5}), \ldots, d_\alpha d_\beta d_\gamma d_\delta \Psi_0 = o(r^{-5}), \\ d_\alpha D\Psi_0 &= o(r^{-6}), \ldots, d_\alpha d_\beta d_\gamma d_\delta D\Psi_0 = o(r^{-6}), \end{aligned} \tag{7.76}$$

where

$$d_\alpha = \frac{\partial}{\partial X^\alpha}, \quad \alpha, \beta, \gamma \ldots = 2, 3.$$

Using the relationships (7.74)-(7.76), we can find the asymptotic behavior of all the other spinor quantities that enter the equations (7.68)-(7.73). For instance, we will show how the asymptotic behavior of the quantities ρ and σ is to be determined. We will introduce the matrices [40]

$$P = \left\{ \begin{array}{cc} \rho & \sigma \\ \overline{\sigma} & \rho \end{array} \right\}, \quad Q = \left\{ \begin{array}{cc} 0 & \Psi_0 \\ \overline{\Psi}_0. & 0 \end{array} \right\}$$

Then the equations (7.69a), (7.69b) and their complex conjugates

$$\begin{aligned} D\rho &= \rho^2 + \sigma\overline{\sigma}, \quad D\rho = \rho^2 + \sigma\overline{\sigma}, \\ D\sigma &= 2\rho\sigma + \Psi_0, \quad D\overline{\sigma} = 2\rho\overline{\sigma} + \overline{\Psi}_0 \end{aligned} \tag{7.77}$$

will become

$$DP = P^2 + Q. \tag{7.78}$$

This equation has a solution of the form

$$P = -(DY)^{-1}, \tag{7.79}$$

where

$$Y = \left(\begin{array}{cc} y_1 & y_2 \\ \overline{y}_1 & \overline{y}_2 \end{array} \right) \tag{7.80}$$

is a nonsingular solution (for a given P) to the equations

$$DY = -PY. \tag{7.81}$$

It is seen from (7.79) and (7.81) that the matrix (7.80) obeys the equations

$$D^2 Y = -QY. \tag{7.82}$$

The asymptotic behavior of the solutions of the equation (7.82) for the case where

$$\int r|\Psi_0| dr = o(1),$$

has the form [40]

$$DY = F + o(1), \tag{7.83}$$

$$Y = rF + o(r), \tag{7.84}$$

where F is a constant matrix. Since in this case $Q = o(r^{-5})$, we will obtain from (7.82) and (7.84)

$$D^2Y = -rQF + o(r^{-4}) = o(r^4). \tag{7.85}$$

Integrating this twice gives

$$DY = F + o(r^{-3}), \tag{7.86}$$

$$Y = rF + E + o(r^{-2}), \tag{7.87}$$

where E is a constant matrix. Solution (7.79) can now be written as

$$P = -r^{-1}I + r^{-2}EF^{-1} + o(r^{-3}). \tag{7.88}$$

Here E is a nonsingular matrix and I is a unit matrix. If $F = 0$, we have from (7.88)

$$\rho = -r^{-1} + o(r^{-2}), \quad \sigma = o(r^{-2}). \tag{7.89}$$

Using other equations of the system (7.68)-(7.73) and working through the same procedure we can find for the quantities in these equations the following asymptotic properties [40]:

$$\begin{array}{cc} \xi^\beta = o(r^{-1}), & \alpha, \beta, \lambda, \mu, \tau = o(r^{-1}), \\ X^\beta, \omega = o(1), & \nu, \gamma = o(1), \\ U = o(r), & \Psi_1 = o(r^{-4}), \end{array} \tag{7.90}$$

$$\Psi_2 = o(r^{-3}), \quad \Psi_3 = o(r^{-2}), \quad \Psi_4 = o(r^{-1}).$$

To get a closer look at the asymptotic behavior of the quantities (7.89) and (7.90) we will do the following [60]. We will write (7.89) as

$$\begin{array}{c} \rho = -r^{-1} + g(r), \\ \sigma = h(r), \end{array} \tag{7.91}$$

where $g, h = o(r^{-2})$.

Substituting (7.91) into (7.69a) and (7.69b) gives

$$\begin{array}{c} Dg + 2r^{-1}g = g^2 + h\overline{h} = o(r^{-4}), \\ Dh + 2r^{-1}h = 2gh + \Psi_0 = o(r^{-4}). \end{array} \tag{7.92}$$

We will at first look for a solution to these equations to within the terms of the order of magnitude $o(r^{-3})$. Integrating the equations, we find $g(r)$ [60]

$$\begin{array}{l} g = e^{-\int 2dr/r} \left\{ \int e^{\int 2dr/r} o(r^{-4}) dr + \rho^0 \right\} = \\ = r^{-2} \left\{ \int o(r^{-4}) dr + \rho^0 \right\} = \rho^0 r^{-2} + o(r^{-3}) \end{array} \tag{7.93}$$

and a similar solution for $h(r)$. In (7.93) the sign 0 on the integration constant ρ implies that the constant is independent of r. Hence

$$\begin{aligned} \rho &= -r^{-1} + \rho^0 r^{-2} + o(r^{-3}), \quad & \rho^0 = \rho^0(u, x^\alpha), \\ \sigma &= \sigma^0 r^{-2} + o(r^{-3}), & \sigma^0 = \sigma^0(u, x^\alpha), \end{aligned} \tag{7.94}$$

$$\alpha = 2, 3.$$

Using the coordinate transformations $r' = r - r^0(u, x^\alpha)$ we can eliminate the term ρ^0/r'^2, therefore

$$\rho = -r^{-1} + o(r^{-3}),$$

$$\sigma = \sigma^0 r^{-2} + o(r^{-3}).$$

Putting again

$$\rho = -r^{-1} + g(r), \quad \sigma = \sigma^0 r^{-2} + h(r),$$

where

$$g(r), h(r) = o(r^{-3}), \tag{7.95}$$

and collecting all the terms in (7.69a) and (7.69b) up to those of the order of magnitude $o(r^{-5})$, we have

$$\begin{aligned} Dg + 2r^{-1}g = o(r^{-4}) = \sigma^0\overline{\sigma}^0 r^{-4} + o(r^{-5}), \\ Dh + 2r^{-1}h = o(r^{-5}). \end{aligned} \tag{7.96}$$

Integrating (7.95) gives

$$g = r^{-2}\left\{\int(\sigma^0\overline{\sigma}^0 r^{-4} + o(r^{-5}))r^2 dr + C_1\right\},$$

$$h = r^{-2}\left\{\int r^2 o(r^{-5}) + C_2\right\}$$

or

$$g = C_1 r^{-2} - \sigma^0\overline{\sigma}^0 r^{-3} + o(r^{-4}),$$

$$h = C_2 r^{-2} + o(r^{-4}).$$

It follows from (7.95) that

$$C_1 = C_2 = 0,$$

therefore

$$\begin{aligned} \rho = -r^{-1} - \sigma^0\overline{\sigma}^0 r^{-3} + o(r^{-4}), \\ \sigma = \sigma^0 r^{-2} + o(r^{-4}). \end{aligned} \tag{7.97}$$

Going over the procedure, we can find that

$$\begin{aligned} \rho = -r^{-1} - \sigma^0\overline{\sigma}^0 r^{-3} + o(r^{-5}), \\ \sigma = \sigma^0 r^{-2} + (\sigma^0\overline{\sigma}^0 - 0,5\Psi_0^0)r^{-4} + o(r^{-5}). \end{aligned} \tag{7.98}$$

Similarly, we can find the asymptotic behavior in r of other variables. The following are the results obtained in [60]:

(a) for spinor components of the Riemannian tensor

$$(7.99)$$

$$\Psi_0 = \Psi_0^0 r^{-5} + o(r^{-6}), \qquad (7.99a)$$

$$\Psi_1 = \Psi_1^0 r^{-4} + (4\alpha^0\Psi_0^0 - \overline{\xi}^{0\alpha}\Psi_{0,\alpha}^0)r^{-5} + o(r^{-6}), \qquad (7.99b)$$

$$\Psi_2 = \Psi_2^0 r^{-3} + (2\alpha^0\Psi_1^0 - \overline{\xi}^{0\alpha}\Psi_{1,\alpha}^0)r^{-4} + o(r^{-5}), \qquad (7.99c)$$

$$\Psi_3 = \Psi_3^0 r^{-2} - \overline{\xi}^{0\alpha}\Psi_{2,\alpha}^0 r^{-3} + o(r^{-4}), \qquad (7.99d)$$

$$\Psi_4 = \Psi_4^0 r^{-3} + (2\alpha^0\Psi_3^0 + \overline{\xi}^{0\alpha}\Psi_{3,\alpha}^0)r^{-2} + o(r^{-3}), \qquad (7.99e)$$

$$\alpha = 2,3;$$

(b) for spinor components of the Ricci rotation coefficients

$$(7.100)$$

$$\rho = -r^{-1} - \sigma^0\overline{\sigma}^0 r^{-3} + o(r^{-5}), \qquad (7.100a)$$

$$\sigma = \sigma^0 r^{-2} + (\sigma^0\overline{\sigma}^0 - 0,5\Psi_0^0)r^{-4} + o(r^{-5}), \qquad (7.100b)$$

$$\alpha = \alpha^0 r^{-1} + \overline{\sigma}^0\overline{\alpha}^0 r^{-2} + \sigma^0\overline{\sigma}^0 r^{-3}\overline{\alpha}^0 + o(r^{-4}), \qquad (7.100c)$$

$$\beta = -\alpha^0 r^{-1} + \sigma^0\alpha^0 r^{-2} - (\sigma^0\overline{\sigma}^0\overline{\alpha}^0 + 0,5\Psi_1^0)r^{-3} + o(r^{-4}), \qquad (7.100d)$$

$$\tau = -0,5r^{-3}\Psi_1^0 + \frac{1}{6}r^{-4}(2\overline{\xi}^{0\alpha}\Psi_{0,\alpha}^0 - 8\alpha^0\Psi_0^0 + \sigma^0\overline{\Psi}_1^0) + o(r^{-5}), \qquad (7.100e)$$

$$\lambda = \lambda^0 r^{-1} - \overline{\sigma}^0\mu^0 r^2 + (0,5\Psi_1^0 + \sigma^0\overline{\sigma}^0\overline{\alpha}^0)r^{-3} + o(r^{-4}), \qquad (7.100f)$$

$$\mu = \mu^0 r^{-1} - (\sigma^0\lambda^0 + \Psi_2^0)r^{-2} + (\sigma^0\overline{\sigma}^0\mu^0 - \alpha^0\Psi_1^0 +$$
$$+ 0,5\overline{\xi}^{0\alpha}\Psi_{1,\alpha}^0)r^{-3} + o(r^{-4}), \qquad (7.100g)$$

$$\gamma = \gamma^0 - 0,5\Psi_2^0 r^{-2} + (\frac{1}{3}\overline{\xi}_{1,\alpha}^0 - \frac{1}{6}\overline{\alpha}^0\Psi_1^0 - 0,5\alpha^0\Psi_1^0) + o(r^{-4}), \qquad (7.100h)$$

$$\nu = \nu^0 - \Psi_3^0 r^{-1} + 0,5\overline{\xi}^{0\alpha}\Psi_{2,\alpha}^0 r^{-2} + o(r^{-3}), \qquad (7.100i)$$

$$\alpha = 2,3;$$

(c) for the components of the Newman-Penrose symbols

$$(7.101)$$

$$U = -(\gamma^0 + \overline{\gamma}^0)r + U^0 - 0,5(\Psi_2^0 + \overline{\Psi}_2^0)r^{-1} + \frac{1}{6}r^{-2}(\overline{\xi}^{0\alpha}\Psi_{1,\alpha}^0 +$$
$$+ \xi^{0\alpha}\overline{\Psi}_{1,\alpha}^0) - 2(\alpha^0\Psi_1^0 + \overline{\alpha}_1^0) + o(r^{-4}), \qquad (7.101a)$$

$$X^\alpha = \frac{1}{6}r^{-3}(\Psi_1^0\overline{\xi}^{0\alpha} + \overline{\Psi}_1^0\xi^{0\alpha}) + o(r^{-4}), \tag{7.101b}$$

$$\xi^\alpha = \xi^{\alpha 0}r^{-1} - \sigma^0\overline{\xi}^{0\alpha}r^{-2} + \sigma^0\overline{\sigma}^0\xi^{\alpha 0}r^{-3} + o(r^{-4}), \tag{7.101c}$$

$$\omega = \omega^0 r^{-1} - r^{-2}(\sigma^0\overline{\omega}^0 + 0,5\Psi_1^0) + o(r^{-3}), \tag{7.101d}$$

$$\alpha = 2,3.$$

To simplify the remaining computations we will make use of the coordinate transformations

$$\begin{array}{ll} r' = r + R(0,2,3) & \text{translations} \\ u' = u, x^{\beta'} = x^\beta, \quad b = 2,3, & \text{of the origin of } r, \end{array} \tag{7.102}$$

$$\begin{array}{ll} r' = r/\dot{\gamma}, \quad u' = \gamma(u) & \text{relabeling of} \\ x^{\beta'} = x^\beta, & \text{hypersurfaces,} \end{array} \tag{7.103}$$

$$\begin{array}{ll} r' = r, u' = u, & \text{relabeling of} \\ x^{\beta'} = x^\beta(0,2,3), & \text{geodesics.} \end{array} \tag{7.104}$$

From the equations (7.27a) and (7.27b) we have

$$g^{\alpha\beta} = -(\xi^\alpha\overline{\xi}^\beta + \overline{\xi}^\alpha\xi^\beta) = -(\xi^{0\alpha}\overline{\xi}^{0\beta} + \overline{\xi}^{0\alpha}\xi^{0\beta})r^{-2} + \ldots$$

$$\alpha,\beta = 2,3. \tag{7.105}$$

Using the coordinate transformations (7.102)-(7.104) we can reduce the metric (7.105) to a conformally flat metric [61,62]. Up to the terms of the order of $o(r^{-3})$, we have here

$$g^{22} = g^{33}, \quad g^{23} = g^{32} = 0. \tag{7.106}$$

Since

$$g^{22} = -2\xi^{02}\overline{\xi}^{02}r^{-2} + o(r^{-3}),$$

$$g^{23} = -(\xi^{02}\overline{\xi}^{03} + \overline{\xi}^{02}\xi^{03})r^{-2} + o(r^{-3}),$$

$$g^{33} = -2\xi^{03}\overline{\xi}^{03}r^{-2} + o(r^{-3}),$$

it follows from the conditions (7.106) that

$$\xi^{02} = -i\xi^{03} = P(u, x^\alpha). \tag{7.107}$$

The remaining coordinate transformations for the variables x^2 and x^3 look like [62]

$$x^{2'} + ix^{3'} = f(x^2 + ix^3, u) \tag{7.108}$$

We will next solve a set of nonradial equations (7.71) and (7.72) in order to express the integration "constants" obtained in solving the radial equations through only two functions σ_0 and P.

By way of example, we will consider the nonradial equation (7.72h)

$$\Delta\rho - \delta\tau = (\gamma + \overline{\gamma} - \overline{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2. \tag{7.109}$$

Using the definitions (7.24) of spinor derivatives, we will write (7.109) as

$$\begin{aligned} &\rho_{,0} + U\rho_{,1} + X^{\alpha}\rho_{,\alpha} - \overline{\omega}\tau_{,1} - \\ &\quad -\overline{\xi}^{\alpha}\tau_{,\alpha} - (\gamma + \overline{\gamma} - \overline{\mu})\rho + \\ &\qquad +2\alpha\tau + \lambda\sigma + \Psi_2 = 0. \end{aligned} \tag{7.110}$$

Substituting here the necessary expressions from the solution (7.99)-(7.101) and differentiating with respect to r, we will equate the factors at various degrees of $1/r$ to zero[1]. As a result, from (7.110), we will get:

(1) the factor at $1/r$ is identically zero;

(2) the factor at $1/r^2$ is $(U^0 - \overline{\mu}^0)$, whence

$$U^0 = \overline{\mu}^0;$$

(3) the factor at $1/r^3$ is

$$(\sigma^0\overline{\sigma}^0)_{,0} + 2\sigma^0\overline{\sigma}^0(\gamma^0 + \overline{\gamma}^0) - (\sigma^0\lambda^0 + \overline{\sigma}^0\overline{\lambda}^0) = 0,$$

and this relationship defines γ^0 and λ^0, if the other terms are known;

(4) the factor at $1/r^4$ is identically zero.

We will introduce the notation

$$\nabla = \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3} \tag{7.111}$$

then the final expressions for the ¡¡constants¿¿ $\alpha^0, \gamma^0, \nu^0, \ldots$ in terms of the two main functions P and σ become:

$$\begin{gathered} \gamma^0 = -0,5(\ln\overline{P})_{,0}, \\ \alpha^0 = 0,5\overline{P\nabla}(\ln P)_{,0}, \\ \nu^0 = -0,5\overline{P\nabla}(\ln P\overline{P})_{,0}, \\ \omega^0 = \overline{P}\left(\overline{\nabla}\sigma^0 - 2\sigma^0\overline{\nabla}(\ln P)\right), \\ \lambda^0 = \overline{\sigma}^0\left[\ln(\sigma^0 P^{1/2}\overline{P}^{-3/2})\right]_{,0}, \\ \mu^0 = U^0 = -0,5P\overline{P}\nabla\overline{\nabla}\ln(P\overline{P}), \end{gathered} \tag{7.112}$$

$$\begin{aligned} &\Psi_2^0 - \overline{\Psi_2^0} = (\overline{P\nabla}\omega^0 + \\ &+2\overline{\alpha}^0\overline{\omega}^0 + \overline{\sigma}^0\overline{\lambda}^0) - \\ &-(P\nabla\overline{\omega}^0 + 2\alpha^0\omega^0 + \sigma^0\lambda^0), \\ &\Psi_3^0 = \overline{P\nabla}\mu^0 - P\nabla\lambda^0 + 4\overline{\alpha}^0\overline{\lambda}^0, \\ &\Psi_4^0 = \overline{P\nabla}\nu^0 + 2\alpha^0\nu^0 - \\ &-\lambda^0_{,0} - 4\gamma^0\lambda^0. \end{aligned} \tag{7.113}$$

The functions $\Psi_2^0 + \overline{\Psi_2^0}$, Ψ_0^0 and Ψ_1^0 in addition to the functions σ^0 and P are the basis functions for insular-type systems.

[1]For instance, if we have the asymptotic expression $Ar^{-1} + Br^{-2} + Cr^{-3} + o(r^{-4}) = 0$ (A, B, C are independent of r), then, multiplying this expression by r and putting $r \to \infty$, we will get $A = 0$. Further, multiplying by r^2 and letting $r \to \infty$, we will have $B = 0$, and so forth.

Propagation of the functions Ψ_2^0, Ψ_0^0 and Ψ_1^0 in the u - direction is defined by a group of equations (7.73). For instance, the equation (7.73a), to within terms of the order of magnitude of $o(r^{-6})$, becomes

$$\begin{aligned} \Psi^0_{0,0} r^{-5} + 5(\gamma^0 + \overline{\gamma}^0)\Psi^0_0 r^{-5} - \\ -P\nabla\Psi^0_1 r^{-5} - \gamma^0\Psi^0_0 r^{-5} - \\ -2\overline{\alpha}^0\Psi^0_1 r^{-5} - 3\sigma^0\Psi^0_2 r^{-5} + o(r^{-6}) = 0, \end{aligned}$$

hence

$$\begin{aligned} \Psi^0_{0,0} + 5(\gamma^0 + \overline{\gamma}^0)\Psi^0_0 - \\ -P\nabla\Psi^0_1 - 4\gamma^0\Psi^0_0 - \\ -2\overline{\alpha}^0\Psi^0_1 - 3\sigma^0\Psi^0_2 = 0. \end{aligned} \tag{7.114}$$

The next two equations of group (7.73) (equations (7.73c) and (7.73d)) give

$$\Psi^0_{1,0} + 2(\gamma^0 + 2\overline{\gamma}^0)\Psi^0_1 - P\nabla\Psi^0_2 - 2\sigma^0\Psi^0_3 = 0, \tag{7.115}$$

$$\Psi^0_{2,0} + 3(\gamma^0 + \overline{\gamma}^0)\Psi^0_2 - P\nabla\Psi^0_3 - 2\overline{\alpha}^0\Psi^0_3 - 2\sigma^0\Psi^0_4 = 0. \tag{7.116}$$

The last equation of group (7.73) is satisfied identically. Function P can be chosen so that

$$P_{,0} = 0,$$

i.e.,

$$P = P(x^2, x^3), \quad \beta = 2, 3. \tag{7.117}$$

Under this condition, the equations (7.112) and (7.113) are simplified dramatically to yield

$$\begin{aligned} &\gamma^0 = 0, \\ &\alpha^0 = 0,5\overline{\nabla}P, \\ &\nu = 0, \\ &\omega^0 = P^3\overline{\nabla}(\sigma^0/P^2), \\ &\lambda^0 = \overline{\sigma}^0_{,0}, \\ &\mu^0 = -P^2\nabla\overline{\nabla}\ln P, \end{aligned} \tag{7.118}$$

$$\begin{aligned} (\Psi^0_2 - \overline{\Psi^0_2}) = \\ = P^2\left[\overline{\nabla}(\omega^0/P) - \nabla(\overline{\omega}^0/P)\right] + \\ +\overline{\sigma}^0\sigma^0_{,0} - \sigma^0\overline{\sigma}^0_{,0}, \\ \Psi^0_3 = -P\overline{\nabla}(P^2\nabla\overline{\nabla}\ln P) - \\ -P^3(\overline{\sigma}^0_{,0}/P^2), \\ \Psi^0_4 = -\overline{\sigma}^0_{,00}. \end{aligned} \tag{7.119}$$

Equations (7.73) now become

$$\begin{aligned} \Psi^0_{0,0} - \nabla(P\Psi^0_1) - 3\sigma^0\Psi^0_2 = 0, \\ \Psi^0_{1,0} - P\nabla\Psi^0_1 - 2\sigma^0\Psi^0_3 = 0, \\ \Psi^0_{2,0} - P^2\nabla(\Psi^0_3/P) - \\ -\sigma^0\overline{\sigma}^0_{,00} = 0. \end{aligned} \tag{7.120}$$

We can now write the final form of the Riemannian metric. The metric of the insular-type system looks like

$$g^{ik} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{11} & g^{12} & g^{13} \\ 0 & g^{12} & g^{22} & g^{23} \\ 0 & g^{13} & g^{23} & g^{33} \end{pmatrix}, \tag{7.121}$$

where

$$\begin{aligned} g^{11} = & -2P^2\left[\frac{\partial^2}{\partial x^{2^2}} + \frac{\partial^2}{\partial x^{3^2}}\right]\ln P - (\Psi_2^0 + \overline{\Psi_2^0})r^{-1} + \\ & +\frac{1}{3}P^2\left[\nabla\left\{\frac{\overline{\Psi}_1^0}{P}\right\} + \overline{\nabla}\left\{\frac{\Psi_1^0}{P}\right\} - \right. \\ & \left. - 6P^4\nabla\left\{\frac{\overline{\sigma}^0}{P^2}\right\}\overline{\nabla}\left\{\frac{\sigma^0}{P^2}\right\}\right]r^{-2} + o(r^{-3}), \end{aligned}$$

$$\begin{aligned} g^{12} &= -r^{-2}\Re(f) + r^{-3}\Re(h) + o(r^{-4}), \\ g^{13} &= -r^{-2}\Im(f) + r^{-3}\Im(h) + o(r^{-4}), \\ P &= P(x^2, x^3), \\ f &= 2P^4\nabla(\overline{\sigma}^0/P^2), \\ h &= 4P\left[\tfrac{1}{3}\Psi_1^0 + P^3\sigma^0\nabla(\overline{\sigma}^0/P^2)\right], \end{aligned} \tag{7.122}$$

$$\begin{aligned} g^{22} &= -2P^2r^{-2} + 2P(\sigma^0 + \overline{\sigma}^0)r^{-3} - \\ & \quad -6\sigma^0\overline{\sigma}^0P^2r^{-4} + o(r^{-5}), \\ g^{23} &= -2iP^2(\sigma^0 - \overline{\sigma}^0)r^{-3} + o(r^{-5}), \\ g^{33} &= -2P^2r^{-2} - 2P(\sigma^0 + \overline{\sigma}^0)r^{-3} - \\ & \quad -6\sigma^0\overline{\sigma}^0P^2r^{-4} + o(r^{-5}). \end{aligned}$$

In matrix (7.121) the component g^{11} can be worked out to within the terms of the order of magnitude of $o(r^{-4})$, and the terms $g^{\alpha\beta}$ $(\alpha, \beta = 2, 3)$ to within $o(r^{-5})$.

If now we specify the initial conditions $\Psi_2^0 + \overline{\Psi_2^0}$, Ψ_0^0, Ψ_1^0, σ^0 and P at infinity, then the problem of the initial values will be overcome.

The zero surface of initial values u_0 is determined by the condition

$$\Psi_0^0 = \lim_{r\to\infty}(\Psi_0 r^5) < \infty.$$

The initial value σ^0 is defined on the world tube at spatial infinity. On the tube we chose

$$\sigma^0 = \lim_{r\to\infty}(\sigma r^2)$$

as an independent function of the variables u, x^2, and x^3. The remaining initial data are taken on a two-dimensional surface at infinity that is determined by the intersection of the zero surface u_0 and the world tube. On that two-dimensional surface we specify

$$\Psi_1^0 = \lim_{r\to\infty}(\Psi_1 r^4), \quad \Psi_2^0 + \overline{\Psi_2^0} = \lim_{r\to\infty} r^3(\Psi_2 + \overline{\Psi}_2)$$

and

$$P^2\delta^{\alpha\beta} = \lim_{r\to\infty}(g^{\alpha\beta}r^2)$$

as functions of x^2 and x^3.

7.5 Classification of solutions to the structural Cartan equations of the A_4 geometry by isometry groups

It was shown in Chapter 6 that the structural Cartan equations of the A_4 geometry can be taken to be gauge equations with gauge groups T_4 and $O(3.1)$. Knowledge of this fact alone is not sufficient for one to be able to tell various specific solutions of the structural Cartan equations from one another by group behavior. This is made possible by the technique of the embedding of A_4 geometries into a flat space E_p of many dimensions ($N > 4$).

We will make the following assumption:

(1) we will take the A_4 space to be a continuous deformation of the Minkowski space $E_4(3.1)$;

(2) we will suppose that A_4 has a minimal flat embedding space $E_p(r, s)$ of dimensionality $p = r + s$, where the signature $r + s$ means r positive and s negative diagonal elements of the metric tensor $\eta_{\mu\nu}$ ($\mu, \nu = 1, 2, \dots, p$) of space $E_p(r, s)$.

Let now X^μ be Cartesian coordinates of the flat space $E_p(r, s)$, and $x^{\underline{\alpha}}$ be Gaussian coordinates based on A_4 that is embedded into $E_p(r, s)$. Here and later in the section the Greek indices assume the values $1, \dots, N$.

We will denote the coordinates of a point in space A_4 by $x^{\underline{i}}$; the coordinates in directions orthogonal to A_4, by $x^{\underline{A}}$. Here and in the section the small-cap Roman indices $i, j, k, \dots$ assume the values 1,2,3,4, and the large-cap Roman indices $A, B, C \dots$ the values $5 \dots N$.

In the embedding the coordinates are transformed as follows:

$$X^\mu = X^\mu(x^{\underline{\alpha}}), \tag{7.123}$$

and the tensors between these two reference frames are transformed using the derivatives

$$\begin{aligned} x^{\underline{\alpha}}_{\mu} &= \frac{\partial x^{\underline{\alpha}}}{\partial X^\mu}, \quad & x^{\mu}_{\underline{\alpha}} &= \frac{\partial x_{\underline{\alpha}}}{\partial X_\mu}, \\ X^{\mu}_{\underline{\alpha}} &= \frac{\partial X^\mu}{\partial x^{\underline{\alpha}}}, \quad & X^{\underline{\alpha}}_{\mu} &= \frac{\partial X_\mu}{\partial x_{\underline{\alpha}}}. \end{aligned} \tag{7.124}$$

Thus, if $\eta_{\mu\nu}$ are Cartesian components of the metric tensor of a space $E_p(r,s)$, then its Gaussian components are

$$g^{\underline{\alpha}\underline{\beta}} = x^{\underline{\alpha}}_{\mu} x^{\underline{\beta}}_{\nu} \eta^{\mu\nu}, g_{\underline{\alpha}\underline{\beta}} = X^{\mu}_{\underline{\alpha}} X^{\nu}_{\underline{\beta}} \eta_{\mu\nu}. \tag{7.125}$$

The inverse relationship are

$$\eta^{\mu\nu} = X^{\mu}_{\underline{\alpha}} X^{\nu}_{\underline{\beta}} g^{\underline{\alpha}\underline{\beta}} \eta_{\mu\nu} = x^{\underline{\alpha}}_{\mu} x^{\underline{\beta}}_{\nu} g_{\underline{\alpha}\underline{\beta}} \tag{7.126}$$

If the A_4 space has an isometry group, this group consists of pseudorotations and reflections $O(r,s)$ of the space $E_p(r,s)$.

Suppose now that we have a coordinate transformation in Cartesian coordinates

$$X'^{\mu} = X^{\mu} + U^{\mu}, \quad U^{\mu} = \varepsilon^{\mu}_{\nu} X^{\nu} \tag{7.127}$$

that is essentially one infinitesimal transformation of the group $O(r,s)$, such that $N(N-1)/2$ of infinitesimal quantities $e^{\mu}{}_{\nu}$ are constant and meet the condition $e^{(\mu\nu)} = 0$. The isometric nature of the transformation (7.127) is manifested by the fact that the Lee derivative with respect to U^{ν} of $\eta^{\mu\nu}$ vanishes

$$\mathcal{L}\eta^{\mu\nu} = U^{(\mu,\nu)} = 0. \tag{7.128}$$

On the other hand, the Gaussian coordinates are transformed as

$$x^{\underline{\alpha}'} = x^{\underline{\alpha}} + \xi^{\underline{\alpha}}, \tag{7.129}$$

where $\xi^{\underline{\alpha}} = x^{\underline{\alpha}} U^{\mu}$ are group generators.

Relationship (7.129) can be split into two parts

$$\begin{array}{ll} x^{\underline{i}'} = x^{\underline{i}} + \xi^{\underline{i}}, & x^{\underline{A}'} = x^{\underline{A}} + \xi^{\underline{A}}, \\ i = 1,2,3,4, & A = 5\ldots p. \end{array} \tag{7.130}$$

The embedded A_4 space in a Gaussian reference frame is now subject to the condition

$$x^{\underline{A}} = 0. \tag{7.131}$$

If $f(x^{\underline{\alpha}})$ is any real function defined in $E_p(r,s)$, then its space-time part will be

$$\times f(x^{\underline{\alpha}}) = f(x^{\underline{\alpha}})|_{A_4} \quad \text{at} \quad x^{\underline{A}} \to 0.$$

A function defined only on an embedded A_4 surface will be denoted as $f(A_4)$. For instance, we will have

$$g^{\underline{ij}}\Big|_{A_4} = x^{\underline{i}}_{\mu} x^{\underline{j}}_{\nu} \eta^{\mu\nu}\Big|_{A_4} = g^{\underline{ij}} A_4.$$

The Killing equations for the vector ξ^{α} look like

$$\mathcal{L} g^{\underline{\alpha}\underline{\beta}} = \overset{*}{\nabla}{}^{(\underline{\beta}} \xi^{\underline{\alpha})} = 0 \tag{7.132}$$

where

$$\overset{*}{\nabla}{}^{(\underline{j}}\xi^{\underline{i})} = 0, \quad \overset{*}{\nabla}{}^{(\underline{A}}\xi^{\underline{i})} = 0, \quad \overset{*}{\nabla}{}^{(\underline{B}}\xi^{\underline{A})} = 0. \tag{7.133}$$

To the transformations (7.130) in an embedded A_4 correspond the transformations

$$x^{\underline{i}'} = x^{\underline{i}} + \xi^{\underline{i}}|_{A_4} = 0, \quad x^{\underline{A}'} = x^{\underline{A}} + \xi^{\underline{A}}|_{A_4} = 0 \tag{7.134}$$

on which the following conditions are imposed:

$$\overset{*}{\nabla}{}^{(\underline{j}}\xi^{\underline{i})}|_{A_4} = 0, \quad \overset{*}{\nabla}{}^{(\underline{A}}\xi^{\underline{i})}|_{A_4} = 0, \quad \overset{*}{\nabla}{}^{(\underline{B}}\xi^{\underline{A})}|_{A_4} = 0. \tag{7.135}$$

The covariant derivative of the vector $\xi_{\underline{\alpha}}$ in a Gaussian reference frame with respect to the connection $\Delta^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$ is

$$\overset{*}{\nabla}_{\underline{\beta}}\, \xi_{\underline{\alpha}} = \xi_{\underline{\alpha},\underline{\beta}} + \Delta^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}\xi_{\underline{\gamma}} = 0. \tag{7.136}$$

It is seen that the expression for $\overset{*}{\nabla}_{\underline{\beta}}\, \xi_{\underline{\alpha}}$ does not coincide with the expression for the covariant derivative in A_4 space, unless the following condition is met:

$$x^{\underline{A}}|_{A_4} = 0. \tag{7.137}$$

Condition (7.137) has the meaning that transformation (7.129) does not change the definition A_4. Condition (7.137) identifies in the group $O(r,s)$ a subgroup that defines the symmetry of the embedded A_4 space. By adding the group $O(r,s)$ reflections and the condition (7.137), we will get the isometry group of A_4 space. Since the maximal dimensionality of an embedded space for Riemannian spaces of dimensionality 4 is 10, then going over the signatures of embedded spaces makes it possible to establish 22 isometric groups [63].

Given in table 7.1 are Lee isometric groups for various specific A_4 spaces and their spinor representations.

The table also provides the most important subgroups. It is sufficient to specify one of the groups in the table to give an isometric definition of the appropriate A_4 geometry. On the other hand, each solution of the structural Cartan equations of the A_4 geometry has corresponding to it an embedded space.

Shown in table 7.2 are some minimum embedded spaces for a series of A_4 spaces that feature various Riemannian metrics [63].

All these spaces can be derived as solutions to the structural Cartan equations of A_4 geometry (e.g., Riemannian metric of Gödel space has been obtained in Ozsvath [50] using the Newman-Penrose method, i.e., as a solution of the structural equations).

7.6 A_4 geometry with a Schwarzschild-type metric

In order to construction an A_4 geometry that has a Schwarzschild metric the following condition must be met: (7.45), (7.49), (7.50), (7.55), (7.59), (7.65),

Table 7.1:

p	$E_p(r,s)$	$L_p(r,s)$	Spinor group	Most important subgroups
4	$E_4(3.1)$	$SO(3.1)$	$SL(2.C)$	
4	$E_4(2.2)$	$O(3.1)$	$SU(1.1)\times SU(1.1)$	
5	$E_5(4.1)$	$SO(4.1)$	$SL(4.C)$	$SU(2)\times SU(2)$
5	$E_5(3.2)$	$SO(3.2)$	$SU(1.1.1.1)$	$SU(1.1)\times SU(1.1)$
6	$E_6(5.1)$	$O(5.1)$	$SL(4.C)$	
6	$E_6(4.2)$	$O(4.2)$	$SU(2.2)$	$SU(2)\times SU(2)$
6	$E_6(3.3)$	$O(3.3)$	$SL(4.C)$	$SU(1.1)\times SU(1.1)$
7	$E_7(6.1)$	$SO(6.1)$	$SL(8.C)$	$SU(4)$
7	$E_7(5.2)$	$SO(5.2)$	$SU(2.2.2.2)$	$SU(2.2)$
7	$E_7(4.3)$	$SO(4.3)$	$SL(8.C)$	$SU(2)\times SU(2)$
8	$E_8(7.1)$	$O(7.1)$	$SL(8.C)$	$SU(4)$
8	$E_8(6.2)$	$O(6.2)$	$SU(1.1)\times SU(4.4)$	$SU(4)$
8	$E_8(5.3)$	$O(5.3)$	$SL(16.C)$	$SU(2)\times SU(2)$
8	$E_8(4.4)$	$O(4.4)$	$SU(1.1)\times SU(2.2.2.2)$	$SU(2)\times SU(2)$
9	$E_9(8.1)$	$SO(8.1)$	$SL(16.C)$	$SU(4)$
9	$E_9(7.2)$	$SO(7.2)$	$SU(4.4.4.4)$	$SU(4.4)$
9	$E_9(6.3)$	$SO(6.3)$	$SL(16.C)$	$SU(4)$
9	$E_9(5.4)$	$SO(5.4)$	$SU(2.2.2.2.2.2.2.2)$	$SU(2)\times SU(2)$
10	$E_{10}(9.1)$	$O(9.1)$	$SL(16.C)$	
10	$E_{10}(8.2)$	$O(8.2)$	$SU(8.8)$	$SU(8)$
10	$E_{10}(7.3)$	$O(7.3)$	$SL(16.C)$	
10	$E_{10}(6.4)$	$O(6.4)$	$SU(4.4.4.4)$	$SU(4)$
10	$E_{10}(5.5)$	$O(5.5)$	$SL(16.C)$	

Table 7.2:

$E_{p(r.s)}$	Metric of immersed space
E (4.1)	De Sitter-Einstein space
E_6 (5.1)	Kruskal space
E_6 (4.2)	Schwarzschild space
E_7 (5.2)	Petrov space $T_2/C4/4$ [22]
E_7 (4.3)	Petrov space $T_1/C4/5,6$
E_9 (6.3)	Robinson-Trautman space $C \leq 0$
E_9 (5.4)	Robinson-Trautman space $C \geq 0$
E_{10} (6.4)	Axial-symmetrical Weyl space
E_{10} (5.5)	Gödel space

(7.66) and (7.67). The physical meaning of these constraints has been considered earlier in the book. As a result the structural Cartan equations (7.69)-(7.73) become:

(1) radial equations that contain a derivative with respect to r

$$\left.\begin{array}{ll}
D\xi^\alpha = \rho\xi^\alpha, & (7.138a) \\
D\omega = \rho\omega - (\overline{\alpha} + \beta), & (7.138b) \\
DX^\alpha = (\overline{\alpha} + \beta)\overline{\xi}^\alpha + (\alpha + \overline{\beta})\overline{\xi}^\alpha, & (7.138c) \\
DU = (\overline{\alpha} + \beta)\omega + (\alpha + \overline{\beta})\omega - (\gamma + \overline{\gamma}), & (7.138d)
\end{array}\right| \qquad (7.138)$$

$$\left.\begin{array}{ll}
D\rho = \rho^2, & (7.139a) \\
0 = 0, & (7.139b) \\
D\tau = \tau\rho, & (7.139c) \\
D\alpha = \alpha\rho, & (7.139d) \\
D\beta = \beta\rho, & (7.139e) \\
D\gamma = \tau\alpha + \overline{\tau}\beta + \Psi_2, & (7.139f) \\
0 = 0, & (7.139g) \\
D\mu = \mu\rho + \Psi_2, & (7.139h) \\
0 = 0, & (7.139i)
\end{array}\right| \qquad (7.139)$$

$$\left.\begin{array}{ll}
0 = 0, & (7.140a) \\
D\Psi_2 = 3\rho\Psi_2, & (7.140b) \\
\overline{\delta}\Psi_2 = 0, & (7.140c) \\
0 = 0; & (7.140d)
\end{array}\right| \qquad (7.140)$$

(2) nonradial equations

$$\left.\begin{array}{ll}
\delta X^\alpha - \Delta\xi^\alpha = (\mu + \overline{\gamma} - \gamma)\xi^\alpha, & (7.141a) \\
\delta\overline{\xi}^\alpha - \overline{\delta}\xi^\alpha = (\overline{\beta} - \alpha)\xi^\alpha + (\overline{\alpha} - \beta)\overline{\xi}^\alpha, & (7.141b) \\
\delta\overline{\omega} - \overline{\delta}\omega = (\overline{\beta} - \alpha)\omega + (\overline{\alpha} - \beta)\overline{\omega} + (\mu - \overline{\mu}), & (7.141t) \\
\delta U - \Delta\omega = (\mu + \overline{\gamma} - \gamma)\omega, & (7.141c)
\end{array}\right| \qquad (7.141)$$

$$\left.\begin{array}{ll}
0 = 0, & (7.142a) \\
\delta\rho = (\beta + \overline{\alpha})\rho, & (7.142b) \\
\delta\alpha - \overline{\delta}\beta = \mu\rho - 2\alpha\beta + \alpha\overline{\alpha} + \beta\beta - \Psi_2, & (7.142c) \\
\overline{\delta}\mu = -(\alpha + \overline{\beta})\mu, & (7.142d) \\
-\Delta\mu = \mu\gamma + \overline{\gamma}\mu + \mu^2, & (7.142e) \\
\delta\gamma - \Delta\beta = \tau\mu + (\mu - \gamma + \overline{\gamma})\beta, & (7.142f) \\
\delta\tau = 2\tau\beta, & (7.142g) \\
\Delta\rho - \overline{\delta}\tau = (\gamma + \overline{\gamma} - \overline{\mu})\rho - 2\alpha\tau - \Psi_2, & (7.142h) \\
\Delta\alpha - \overline{\delta}\gamma = (\overline{\gamma} - \gamma - \overline{\mu})\alpha; & (7.142i)
\end{array}\right| \qquad (7.142)$$

(3) U-derivative equations

$$\left.\begin{array}{ll}
0 = 0, & (7.143a) \\
\delta\Psi_2 = 3\tau\Psi_2, & (7.143b) \\
\Delta\Psi_2 = -3\mu\Psi_2, & (7.143c) \\
0 = 0. & (7.143d)
\end{array}\right| \qquad (7.143)$$

In the reference frame we have chosen the structural equations have the form

$$D = \frac{\partial}{\partial r}, \quad \delta = \omega\frac{\partial}{\partial r} + \xi^{\beta}\frac{\partial}{\partial x^{\beta}},$$
$$\Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^{\beta}\frac{\partial}{\partial x^{\beta}}, \quad \overline{\delta} = \overline{\omega}\frac{\partial}{\partial r} + \overline{\xi}^{\beta}\frac{\partial}{\partial x^{\beta}},$$
$$\beta = 2, 3.$$

and the commutation relation

$$\nabla_{[a}\nabla_{b]} = -\Omega_{ab}^{\cdot\cdot c} \tag{7.144}$$

become

$$\tag{7.145}$$

$$\Delta D - D\Delta = (\gamma + \overline{\gamma})D - (\overline{\tau} + \pi)\delta - (\tau + \overline{\pi})\overline{\delta}, \tag{7.145a}$$

$$\delta D - D\delta = (\overline{\alpha} + \beta - \overline{\pi})D - \overline{\rho}\delta, \tag{7.145b}$$

$$\delta\Delta - \Delta\delta = (\tau - \overline{\alpha} - \beta)\Delta + (\mu - \gamma + \overline{\gamma})\delta, \tag{7.145c}$$

$$\overline{\delta}\delta - \delta\overline{\delta} = (\overline{\mu} + \mu)D + (\overline{\rho} - \rho)\Delta - (\overline{\beta} - \alpha)\delta - (\overline{\alpha} - \beta)\overline{\delta}, \tag{7.145d}$$

therefore we can proceed with integration of the equations. Integration begins with the radial equations that contain the derivative D. For instance, the solution to the equation (7.139a), with the condition $\rho = \overline{\rho}$ has the form

$$\rho = -r^{-1}. \tag{7.146}$$

Differentiating (7.139a) with respect to $\overline{\delta}$ gives

$$\overline{\delta}D\rho = 2\rho\overline{\delta}\rho. \tag{7.147}$$

Applying the complex-conjugate operator (7.145b) to ρ, we have ($\pi = 0$)

$$(\overline{\delta}D - D\overline{\delta})\rho = (\alpha + \overline{\beta})D\rho - \rho\overline{\delta}\rho, \tag{7.148}$$

whend by (7.147), we have

$$D\overline{\delta}\rho - 3\rho\overline{\delta}\rho = -\rho^2(\alpha + \overline{\beta}). \tag{7.149}$$

Using (7.139a), (7.139d) and (7.139e), we obtain the general solution (7.149)

$$\overline{\delta}\rho = \rho(\alpha + \overline{\beta}) - 2\overline{\tau}^0\rho^3, \tag{7.150}$$

where $\overline{\tau}^0$ is an integration constant. Calculating $(\Delta D - D\Delta)\Psi$, $(\delta\Delta - \Delta\delta)\Psi$ and $(\overline{\delta}\delta - \delta\overline{\delta})\Psi$ (here and later we will use the notation $\Psi_2 = \Psi$, $\Psi_2^0 = \Psi^0$) using the relationships (7.140b,c), (7.143b,c) and (7.145), we will arrive at three new equalities

$$\Delta\rho + D\mu = \rho(\gamma + \overline{\gamma}) - \tau\overline{\tau}, \tag{7.151}$$

$$\overline{\delta}\rho = \rho(\alpha + \overline{\beta}), \tag{7.152}$$

$$\delta\mu + \Delta\tau = -\mu(\overline{\alpha} + \beta) + \tau(\gamma - \overline{\gamma}). \tag{7.153}$$

Substituting (7.150) into (7.152) and integrating, we will get $\overline{\tau}^0\rho^2 = 0$, whence $\overline{\tau}^0 = \tau^0 = 0$.

Integrating the remaining radial equations gives [65]

$$\begin{aligned}
\beta &= \rho\beta^0, \\
\alpha &= \rho\alpha^0 + \rho^2\overline{\tau}, \\
\tau &= \rho\eta^0, \\
\gamma &= \gamma^0 + \rho\eta^0\alpha^0 + \rho\overline{\eta}^0\beta^0 + \rho^2 1/2\Psi^0, \\
\Psi &= \rho^3\Psi^0, \\
U &= U^0 - r(\gamma^0 + \overline{\gamma}^0) + \rho\left\{\eta^0\overline{\omega}^0 - 1/2\Psi^0\right\} + \rho\left\{\overline{\eta}^0\omega^0 - 1/2\overline{\Psi}^0\right\}, \\
\omega &= \rho\omega^0 + \overline{\alpha}^0 + \beta^0, \\
\xi^\alpha &= \rho\xi^{\alpha 0}, \\
X^\alpha &= X^{0\alpha} + \rho\eta^0\xi^{0\alpha} + \rho\overline{\eta}^0\overline{\xi}^{0\alpha}, \quad \alpha = 0, 2, 3.
\end{aligned} \tag{7.154}$$

Applying the operator (7.145a) to ρ and using the equation (7.139a), we will obtain

$$2\rho\Delta\rho - D\Delta\rho = \rho^2(\gamma + \overline{\gamma}) - \tau\overline{\delta}\rho - \overline{\tau}\delta\rho. \tag{7.155}$$

Solving this equation using (7.142b), (7.150), (7.154) and considering that $\tau^0 = 0$, we get

$$\begin{aligned}
\Delta\rho = & -M^0\rho^2 + \eta^0(\alpha^0 + \overline{\beta}^0)\rho^2 + (\gamma^0 + \overline{\gamma}^0)\rho + \\
& + \left[\overline{\eta}^0(\overline{\alpha}^0 + \beta^0) - \eta^0\overline{\eta}^0\right]\rho^2 - 1/2\rho^3(\Psi^0 + \overline{\Psi^0}).
\end{aligned} \tag{7.156}$$

Substituting this relation into (7.151) and integrating gives

$$\mu = \mu^0 + \rho M^0 + 1/2\rho^2(\Psi^0 + \overline{\Psi}^0). \tag{7.157}$$

The next phase of integration consists in substituting the derived solutions of the radial equations (7.146), (7.154) and (7.155) into the remaining unused equations. After differentiating with respect to r, we will equate to zero the factors at the same degrees of $1/r$. We will end up with a set of equations for quantities independent of r.

Applying the operators δ, $\overline{\delta}$ and Δ to $\rho = -1/r$, we have

$$\delta\rho = \omega\rho^2, \quad \overline{\delta}\rho = \overline{\omega}\rho^2, \quad \Delta\rho = U\rho^2. \tag{7.158}$$

Comparing these equalities with (7.142b), (7.150) and (7.156) gives

$$M^0 = \overline{M}^0, \tag{7.159}$$

$$\omega^0 = 0, \tag{7.160}$$

$$U^0 = \eta^0(\alpha^0 + \overline{\beta}^0) + \overline{\eta}^0(\overline{\alpha}^0 + \beta^0) - \eta^0\overline{\eta}^0 - M^0. \tag{7.161}$$

From (7.142c) we get

$$\mu^0 = 0, \tag{7.162}$$

$$\overline{\xi}^{0\alpha}\tau^0_{,\alpha} = -\tau^0(\alpha^0 + 3\overline{\beta}^0 - \eta^0) + 1/2(\Psi^0 - \overline{\Psi}^0), \tag{7.163}$$

$$\overline{\xi}^{0\alpha}\eta^0_{,\alpha} = 2\overline{\beta}^0\eta^0. \tag{7.164}$$

Consequently, from the equations (7.142g), (7.142d) and (7.142c) we will have

$$\begin{aligned} \xi^{0\alpha}\tau^0_{,\alpha} &= -\tau^0(3\overline{\alpha}^0 + \beta^0), \\ \xi^{0\alpha}\eta^0_{,\alpha} &= -\eta^0(2\overline{\alpha}^0 - \eta^0), \end{aligned} \tag{7.165}$$

$$\begin{aligned} \xi^{0\alpha}\alpha^0_{,\alpha} - \overline{\xi}^{0\alpha}\beta^0_{,\alpha} &= 2\beta^0(\overline{\beta}^0 - \alpha^0) + M^0, \\ \overline{\xi}^{0\alpha}M^0_{,\alpha} &= -2M^0(\alpha^0 + \overline{\beta}^0), \\ \eta^0 &= 0. \end{aligned} \tag{7.166}$$

Substitution of the last of these equations into the equality (7.161) gives

$$U^0 = -M^0. \tag{7.167}$$

From (7.142f-i) and (7.153) we get

$$\begin{aligned} X^{0\alpha}\tau^0_{,\alpha} &= -\tau^0(\gamma^0 + 3\overline{\gamma}^0), \\ \xi^{0\alpha}M^0_{,\alpha} &= -2M^0(\overline{\alpha}^0 + \beta^0), \\ X^{0\alpha}\alpha^0_{,\alpha} - \overline{\xi}^{0\alpha}\gamma^0_{,\alpha} &= -\gamma^0(\alpha^0 - \overline{\beta}^0), \\ X^{0\alpha}\beta^0_{,\alpha} - \xi^{0\alpha}\gamma^0_{,\alpha} &= \gamma^0(\beta^0 + \overline{\alpha}^0) - 2\overline{\gamma}^0\beta^0, \\ X^{0\alpha}M^0_{,\alpha} &= -2M^0(\overline{\alpha}^0 + \overline{\gamma}^0). \end{aligned}$$

Equations (7.141) enables us to write

$$\xi^{0\alpha}X^{0\beta}_{,\alpha} - X^{0\alpha}\xi^{0\beta}_{,\alpha} = 2\overline{\gamma}^0\xi^{0\beta} - (\overline{\alpha}^0 + \beta^0)X^{0\beta},$$

$$\overline{\xi}^{0\alpha}\xi^{0\beta}_{,\alpha} - \xi^{0\alpha}\overline{\xi}^{0\beta}_{,\alpha} = -2\overline{\beta}^0\xi^{0\beta} + 2\beta^0\overline{\xi}^{0\beta},$$

$$\alpha, \beta = 0, 2, 3.$$

Substituting $\Psi = \rho^3\Psi^0$ into the equations (7.140c), (7.143b) and (7.143c), we will obtain

$$\xi^{0\alpha}\Psi^0_{,\alpha} = -3\Psi^0(\overline{\alpha}^0 + \beta^0 - \eta^0), \tag{7.168}$$

$$\overline{\xi}^{0\alpha}\Psi^0_{,\alpha} = -3\Psi^0(\alpha^0 + \overline{\beta}^0). \tag{7.169}$$

$$X^{0\alpha}\Psi^0_{,\alpha} = -3\Psi^0(\gamma^0 + \overline{\gamma}^0 + \mu^0), \tag{7.170}$$

Let us consider the case of

$$\Psi^0 = \overline{\Psi}^0, \quad \Psi^0 = \text{const.} \tag{7.171}$$

Then from (7.166) and (7.167) and also from (7.164) and (7.160)) it follows that

$$\gamma^0 + \overline{\gamma}^0 = 0, \quad \alpha^0 + \overline{\beta}^0 = 0. \tag{7.172}$$

We will constrain ourselves to the case $\tau^0 = 0$. Considering that $\eta^0 = 0$, we will obtain that $\tau = 0$. Equations (7.141) will then become

(7.173)

$$X^{0\alpha} M^0_{,\alpha} = 0, \tag{7.173a}$$

$$\xi^{0\alpha} M^0_{,\alpha} = 0, \tag{7.173b}$$

$$X^{0\alpha}\alpha^0_{,\alpha} - \xi^{0\alpha}\gamma^0_{,\alpha} = -2\gamma^0\alpha^0, \tag{7.173c}$$

$$\xi^{0\alpha}\alpha^0_{,\alpha} - \overline{\xi}^{0\alpha}\beta^0_{,\alpha} = 4\alpha^0\overline{\alpha}^0 + M^0, \tag{7.173d}$$

$$\xi^{0\alpha}X^{0\beta}_{,\alpha} - X^{0\alpha}\xi^{0\beta}_{,\alpha} = 2\overline{\gamma}^0\xi^{0\beta}, \tag{7.173e}$$

$$\overline{\xi}^{0\alpha}\xi^{0\beta}_{,\alpha} - \xi^{0\alpha}\overline{\xi}^{0\beta}_{,\alpha} = -2\overline{\beta}^0\xi^{0\beta} + 2\beta^0\overline{\xi}^{0\beta}. \tag{7.173f}$$

Next we carry out the transformations

$$x^\alpha = x^{\alpha'}(x^\alpha).$$

We will thus have the relation $X^{0\alpha} = \delta^\alpha{}_0$ satisfied. Now the only arbitrary element in the selection of coordinates is the transformations

(7.174)

$$x^{0'} = x^0 + f(x^2, x^3), \tag{7.174a}$$

$$x^{2'} = g(x^2, x^3), \tag{7.174b}$$

$$x^{3'} = h(x^2, x^3). \tag{7.174c}$$

Integrating (7.173) gives

$$M^0 = \text{const.}$$

Using the transformations

$$l^{i'} = l^i, \quad n^{i'} = n^i, \quad m^{i'} = m^i \exp[i\theta^0(x^\alpha)], \tag{7.175}$$

we can achieve

$$\gamma^0 = 0. \tag{7.176}$$

It follows from this and (7.173e) that $\xi^{0\alpha}$ is independent of x^0.

We will introduce the notation

$$\xi^{02} = P(x^2, x^3), \quad \xi^{03} = iP(x^2, x^3) \tag{7.177}$$

and use the transformations (7.175) to make P a real function. The only arbitrariness will be the transformation of the coordinates and the components of the light tetrad of the form

$$l^{i'} = [A^0(x^\alpha)]^{-1} l^i, \quad n^{i'} = A^0(x^\alpha) n^i, \quad m^{i'} = m^i, \\ r' = A^0(x^\alpha) r \tag{7.178}$$

with the constant A^0 in the transformation (7.174a) and the transformation

$$\zeta = g(\zeta),$$

where

$$\zeta = x^2 + ix^3. \tag{7.179}$$

Using the notation (7.111), we will write the equations (7.173d) and (7.173f) in the form

$$\alpha^0 = 1/2 \nabla P, \quad \nabla \left[\frac{\overline{\xi}^{00}}{P} \right] = \overline{\nabla} \left[\frac{\xi^{00}}{P} \right],$$

$$\varepsilon^0 = M^0 = 2P^2 \nabla \overline{\nabla} \ln(\sqrt{2} P) \frac{1}{2}.$$

Using the remaining arbitrariness and the selection of the coordinates and components of the light tetrad enables us to write the solutions of these equations as

$$\sqrt{2} P = 1 + 1/2 \varepsilon^0 \zeta \overline{\zeta}, \quad \xi^{00} = 0, \quad \alpha^0 = 1/2 \varepsilon^0 \zeta,$$

$$\varepsilon^0 = \pm 1/2, 0.$$

We now use the results obtained for the components of the Newman-Penrose symbols to obtain

$$\sigma^i_{0\dot{0}} = (0, 1, 0, 0), \quad \sigma^i_{1\dot{1}} = (1, U, 0, 0), \quad \sigma^i_{0\dot{1}} = \rho(0, 0, P, iP),$$

$$\sigma_i^{0\dot{0}} = (1, 0, 0, 0), \quad \sigma_i^{1\dot{1}} = (-U, 1, 0, 0), \quad \sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0, 0, 1, i).$$

Using the relationship

$$g_{ij} = \varepsilon_{AC} \varepsilon_{\dot{B}\dot{D}} \sigma^{A\dot{B}}_i \sigma^{C\dot{D}}_j$$

we can now derive the metric tensor g_{ik}

$$g_{ik} = \begin{pmatrix} -2U & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -(2\rho^2 P^2)^{-1} & 0 \\ 0 & 0 & 0 & (2\rho^2 P^2)^{-1} \end{pmatrix}, \tag{7.180}$$

where

$$U = -\varepsilon^0 + \Psi^0/r. \tag{7.181}$$

$$\tag{7.182}$$

Let now $e^0 = 1/2$, it is then convenient to go over to the coordinates

$$ct = x^0 - \int dr/2U, \quad r = x^1,$$

$$\sin\theta = \frac{(\zeta\overline{\zeta})^{1/2}}{(1 + 1/4\zeta\overline{\zeta})}, \quad \mathrm{tg}\varphi = \frac{x^3}{x^2}.$$

We will end up with the Riemannian metric

$$ds^2 = \left(1 - \frac{2\Psi^0}{r}\right) c^2 dt^2 - \left(1 - \frac{2\Psi^0}{r}\right)^{-1} dr^2 - \tag{7.183}$$

$$-r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

that coincides with the metric of the Schwarzschild space at

$$\Psi^0 = MG/c^2. \tag{7.184}$$

Notice that, unlike the Schwarzschild metric of Einstein's theory, the metric (7.183) is defined on a translations group T_4 of the geometry of absolute parallelism.

At $\epsilon^0 = 0$ and $\epsilon^0 = -1/2$ we have two more solutions that describe spherically symmetrical objects with mass M (not necessary rest mass), which move at light and faster-than-light velocities

$$ds^2 = \left(-\frac{2\Psi^0}{r}\right) c^2 dt^2 - \left(-\frac{2\Psi^0}{r}\right)^{-1} dr^2 - \tag{7.185}$$

$$-r^2(d\theta^2 + \theta^2 d\varphi^2),$$

$$ds^2 = \left(-1 - \frac{2\Psi^0}{r}\right) c^2 dt^2 - \left(-1 - \frac{2\Psi^0}{r}\right)^{-1} dr^2 - \tag{7.186}$$

$$-r^2(d\theta^2 + \mathrm{sh}^2\theta d\varphi^2).$$

Combining all the results, we will write

Main geometrical characteristics of A_4 geometry with a Riemannian metric of Schwarzschildian type

$$\tag{7.187}$$

1. Coordinate u, r, x^2 and x^3 are given by (7.19).
2. Components of the Newman-Penrose symbols

$$\sigma^{i}_{0\dot{0}} = (0,1,0,0), \quad \sigma^{i}_{1\dot{1}} = (1,U,0,0), \quad \sigma^{i}_{0\dot{1}} = \rho(0,0,P,iP),$$

$$\sigma_{i}^{0\dot{0}} = (1,0,0,0), \quad \sigma_{i}^{1\dot{1}} = (-U,1,0,0), \quad \sigma_{i}^{0\dot{1}} = -\frac{1}{2\rho P}(0,0,1,i),$$

$$U = -1/2 + \Psi^0/r, \quad P = (2)^{-1/2}(1+\zeta\overline{\zeta}/4), \quad \zeta = x^2 + ix^3,$$

$$\Psi^0 = \mathrm{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -1/r, \quad \alpha = -\overline{\beta} = -\alpha^0/r, \quad \gamma = \Psi^0/2r,$$

$$\mu = -\varepsilon^0/r + 2\Psi^0/r^2, \alpha = \zeta/4.$$

4. Spinor components of the Riemannian tensor

$$\Psi = -\Psi^0/r^3.$$

Substituting the components of the Ricci rotation coefficients of the solution (7.187) into the rotational Killing-Cartan metric, we obtain

$$d\tau^2 = -\frac{(\Psi^0)^2}{2r^4}dx_0^2 - \frac{2(\Psi^0 - r)}{r}d\theta^2 - \\ -\frac{2(\Psi^0 - r)\sin^2\theta}{r}d\varphi^2. \tag{7.188}$$

7.7 Some physically meaningful solutions of the structural Cartan equations of A_4 geometry

Skipping detailed computations, we will simply provide some exact solutions of the structural Cartan equations of the A_4 geometry, which are given physical interpretation in the theory of physical vacuum.

7.7.1 Solution with a variable source function

(7.189)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^{i}_{0\dot{0}} = (0,1,0,0), \quad \sigma^{i}_{1\dot{1}} = (1,U,0,0), \quad \sigma^{i}_{0\dot{1}} = \rho(0,0,P,iP),$$

$$\sigma_i^{0\dot{0}} = (1,0,0,0), \quad \sigma_i^{1\dot{1}} = (-U,1,0,0), \quad \sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0,0,1,i),$$

$$U(u) = -1/2 + \Psi^0(u)/r, \quad P = (2)^{-1/2}(1+\zeta\overline{\zeta}/4), \quad \zeta = x^2 + ix^3,$$

$$\Psi^0 = \Psi^0(u).$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -1/r, \quad \alpha = -\overline{\beta} = -\alpha^0/r, \quad \gamma = \Psi^0(u)/2r^2,$$

$$\mu = -1/2r + \Psi^0(u)/r^2, \quad \alpha^0 = \zeta/4.$$

4. Spinor components of the Riemannian tensor

$$\Psi_2 = \Psi = -\Psi^0(u)/r^3, \quad \Phi_{22} = \Phi = -\dot{\Psi}^0(u)/r^2 = -\frac{\partial \Psi^0}{\partial u}\frac{1}{r^2}.$$

The Riemann metric of the solution (7.189) in the coordinates (7.182) has the form

$$ds^2 = \left(1 - \frac{2\Psi^0(t)}{r}\right)c^2dt^2 - \left(1 - \frac{2\Psi^0(t)}{r}\right)^{-1}dr^2 - \qquad (7.190)$$

$$-r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

7.7.2 Solution with quark interaction

(7.191)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = (1,U,0,0), \quad \sigma^i_{0\dot{1}} = \rho(0,0,P,iP),$$

$$\sigma_i^{0\dot{0}} = (1,0,0,0), \quad \sigma_i^{1\dot{1}} = (-U,1,0,0), \quad \sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0,0,1,i)$$

$$U = -1/2 + \tilde{\Lambda}r^2, \quad P = (2)^{-1/2}(1+\zeta\overline{\zeta}/4), \quad \zeta = x^2 + ix^3.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -1/r, \quad \alpha = -\overline{\beta} = -\alpha^0/r, \quad \gamma = \tilde{\Lambda}r,$$

$$\mu = -1/2r - \tilde{\Lambda}\rho r^2, \quad \alpha^0 = \zeta/4.$$

4. Spinor components of the Riemannian tensor

$$\Lambda = \tilde{\Lambda}/6 = R/24 = \text{const}.$$

The metric of the Riemannian solution (7.191) in the coordinates (7.182) has the form

$$ds^2 = \left(1 - \Lambda r^2/3\right)c^2dt^2 - \left(1 - \Lambda r^2/3\right)^{-1}dr^2 - \qquad (7.192)$$

$$-r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

7.7.3 Solution with a short-range (nuclear) interaction

(7.193)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = (1,U,0,0), \quad \sigma^i_{0\dot{1}} = \overline{\rho}(-ir_N\zeta/\sqrt{2},0,P,iP), \quad \sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}}$$

$$\sigma_i^{0\dot{0}} = (1, 0 - r_N x^3/\sqrt{2}\ P,\ r_N x^2/\sqrt{2}P), \quad \sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0,0,1,i), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$

$$\sigma_i^{1\dot{1}} = (-U, 1, U r_N x^3/\sqrt{2}\ P, -U r_N x^2/\sqrt{2}P).$$

$$U = -\frac{1}{2} + \rho\overline{\rho} r_N^2, \quad P = (2)^{-1/2}(1 + \zeta\overline{\zeta}/4), \quad \zeta = x^2 + ix^3,$$

$$r_N = \mathrm{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r + ir_N)^{-1}, \quad \alpha = \rho\alpha^0, \quad \beta = -\overline{\alpha}, \quad \alpha^0 = \zeta/4,$$

$$\gamma = \rho^2\Psi^0/2, \quad \mu = \rho/2 + \rho^2\Psi^0/2 + \rho\overline{\rho}\overline{\Psi}^0/2, \quad \Psi^0 = ir_N.$$

4. Spinor components of the Riemannian tensor

$$\Psi_2 = \Psi = \Psi^0\rho^3.$$

The metric of the Riemannian solution (7.193) in the coordinates (7.182) has the form

$$ds^2 = \Phi[cdt + 4r_N\sin^2(\theta/2)d\varphi]^2 + dr^2/\Phi - \qquad (7.194)$$
$$-(r^2 + r_N^2)(d\theta^2 + \sin^2\theta d\varphi),$$

where

$$\Phi = 1 - \frac{2r_N^2}{r^2 + r_N^2}. \qquad (7.195)$$

7.7.4 Solution with an electronuclear interaction

(7.196)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = (1,U,0,0),$$

$$\sigma^i_{0\dot{1}} = \overline{\rho}(-ir_N\zeta/\sqrt{2},0,P,iP), \quad \sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}}$$

$$\sigma_i^{0\dot{0}} = (1, 0 - r_N x^3/\sqrt{2}\,P,\ r_N x^2/\sqrt{2}P),$$

$$\sigma_i^{1\dot{1}} = (-U, 1, U r_N x^3/\sqrt{2}\,P, -U r_N x^2/\sqrt{2}P),$$

$$\sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0, 0, 1, i), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$

$$U = -\frac{1}{2} + \rho\overline{\rho}(r r_e/2 + r_N^2), \quad P = (2)^{-1/2}(1 + \overline{\zeta}\zeta/4),$$

$$\zeta = x^2 + ix^3, \quad r_N = \mathrm{const}, r_e = \mathrm{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r + ir_N)^{-1}, \quad \alpha = \rho\alpha^0, \quad \beta = -\overline{\alpha}, \quad \alpha^0 = \zeta/4,$$

$$\gamma = \rho^2\Psi^0/2, \quad \mu = \rho/2 + \rho^2\Psi^0/2 + \rho\overline{\rho}\overline{\Psi}^0/2, \quad \Psi^0 = r_e/2 + ir_N.$$

4. Spinor components of the Riemannian tensor

$$\Psi_2 = \Psi = \Psi^0\rho^3.$$

The Riemannian metric of the solution (7.196) in the coordinates (7.182) has the form

$$ds^2 = \Phi[cdt + 4r_N\sin^2(\theta/2)d\varphi]^2 + dr^2/\Phi - \tag{7.197}$$
$$-(r^2 + r_N^2)(d\theta^2 + \sin^2\theta d\varphi),$$

where

$$\Phi = 1 - \frac{r r_e + 2r_N^2}{r^2 + r_N^2}. \tag{7.198}$$

7.7.5 Solution with electronuclearquark interaction

(7.199)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma_{0\dot{0}}^i = (0, 1, 0, 0), \quad \sigma_{1\dot{1}}^i = (1, U, 0, 0),$$

$$\sigma_{0\dot{1}}^i = \overline{\rho}(-ir_N\zeta/\sqrt{2}, 0, P, iP), \quad \sigma_{1\dot{0}}^i = \overline{\sigma_{0\dot{1}}^i}$$

$$\sigma_i^{0\dot{0}} = (1, 0 - r_N x^3/\sqrt{2}\,P,\ r_N x^2/\sqrt{2}P),$$

$$\sigma_i^{1\dot{1}} = (-U, 1, U r_N x^3/\sqrt{2}\,P, -U r_N x^2/\sqrt{2}P),$$

$$\sigma_i^{0\dot{1}} = -\frac{1}{2\rho P}(0, 0, 1, i), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$

$$U = -\frac{1}{2} + \rho\overline{\rho}(r_e r/2 + r_N^2 - 8\tilde{\Lambda}r_N^4) + \tilde{\Lambda}(r^2 + 5r_N^2),$$

$$\zeta = x^2 + ix^3, \quad P = (2)^{-1/2}(1 + \zeta\overline{\zeta}/4),$$

$$r_N = \text{const}, r_e = \text{const}, \tilde{\Lambda} = \text{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r + ir_N)^{-1}, \quad \alpha = \rho\alpha^0, \quad \beta = -\overline{\alpha}, \quad \alpha^0 = \zeta/4,$$

$$\gamma = \gamma^0 + \rho^2\Psi^0/2 - \tilde{\Lambda}r, \quad \mu = \rho/2 + \rho^2\Psi^0/2 + \rho\overline{\rho}\overline{\Psi}^0/2 - \tilde{\Lambda}r^2\rho,$$

$$\Psi^0 = r_e/2 + ir_N = \text{const}, \gamma^0 = -i\tilde{\Lambda}r_N.$$

4. Spinor components of the Riemann tensor

$$\Psi_2 = \Psi = \Psi^0\rho^3, \Lambda = \tilde{\Lambda}/6 = R/24 = \text{const}.$$

The metric of the Riemann solution (7.199) in the coordinates (7.182) has the form

$$ds^2 = \Phi[cdt + 4r_N\sin^2(\theta/2)d\varphi]^2 + dr^2/\Phi - \tag{7.200}$$

$$-(r^2 + r_N^2)(d\theta^2 + \sin^2\theta d\varphi),$$

where

$$\Phi = 1 - \frac{rr_e + 2r_N^2 - 16\tilde{\Lambda}r_N^4}{r^2 + r_N^2} - 2\tilde{\Lambda}(r^2 + 5r_N^2). \tag{7.201}$$

7.7.6 Solution with Coulomb-Newton interaction and three-dimensional rotation of a source

(7.202)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = \rho\overline{\rho}(\Omega, -Y, 0, a),$$

$$\sigma^i_{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}(ia\sin\theta, 0, 1, i\text{cosec}\theta), \quad \sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}},$$

$$\sigma_i^{0\dot{0}} = (1, 0, 0, -a\sin^2\theta),$$

$$\sigma_i^{1\dot{1}} = \rho\overline{\rho}\left(Y, (\rho\overline{\rho})^{-1}, 0, -a\sin^2\theta Y\right),$$

$$\sigma_i^{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}\left(ia\sin\theta, 0, -(\rho\overline{\rho})^{-1}, -i\Omega\sin\theta\right), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$

$$\Omega = r^2 + a^2, \quad Y = (r^2 + a^2 - 2\Psi^0 r)/2,$$

$$a = \text{const}, \quad \Psi^0 = r_e/2 = \text{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r - ia\cos\theta)^{-1}, \beta = -\cot\theta\overline{\rho}/(2)^{3/2},$$
$$\pi = ia\sin\theta\rho^2/(2)^{1/2}, \quad \alpha = \pi - \overline{\beta}, \quad \mu = Y\rho^2\overline{\rho},$$
$$\gamma = \mu + (r + \Psi^0)\rho\overline{\rho}/2, \quad \tau = ia\sin\theta\rho\overline{\rho}/(2)^{1/2}.$$

4. Spinor components of the Riemann tensor

$$\Psi_2 = \Psi = \Psi^0\rho^3.$$

The metric of the Riemannian solution (7.202) in the coordinates (7.182) has the form

$$ds^2 = \left(1 - \frac{2\Psi^0 r}{r^2 + a^2\cos^2\theta}\right)c^2dt^2 + \frac{4\Psi^0 ra}{r^2 + a^2\cos^2\theta}\sin^2\theta d\varphi cdt -$$
$$-\frac{r^2 + a^2\cos^2\theta}{r^2 - 2\Psi^0 r + a^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 -$$
$$-\left(r^2 + a^2 + \frac{2\Psi^0 ra^2}{r^2 + a^2\cos^2\theta}\sin^2\theta\right)\sin^2\theta d\varphi^2. \quad (7.203)$$

7.7.7 Purely torsional solution

(7.204)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = \rho\overline{\rho}(\Omega, -Y, 0, a), \quad \sigma^i_{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}(ia\sin\theta, 0, 1, i\mathrm{cosec}\theta),$$
$$\sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}}, \quad \sigma_i^{0\dot{0}} = (1,0,0,-a\sin^2\theta), \quad \sigma_i^{1\dot{1}} = \rho\overline{\rho}\left(Y, (\rho\overline{\rho})^{-1}, 0, -a\sin^2\theta Y\right),$$
$$\sigma_i^{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}\left(ia\sin\theta, 0, -(\rho\overline{\rho})^{-1}, -i\Omega\sin\theta\right), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$
$$\Omega = r^2 + a^2, \quad Y = (r^2 + a^2)/2, \quad a = \mathrm{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r - ia\cos\theta)^{-1}, \beta = -\cot\theta\overline{\rho}/(2)^{3/2},$$
$$\pi = ia\sin\theta\rho^2/(2)^{1/2}, \quad \alpha = \pi - \overline{\beta}, \quad \mu = Y\rho^2\overline{\rho},$$
$$\gamma = \mu + r\rho\overline{\rho}/2, \quad \tau = ia\sin\theta\rho\overline{\rho}/(2)^{1/2}.$$

The metric of the Riemannian solution (7.204) in the coordinates (7.182) has the form

$$ds^2 = c^2dt^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr - (r^2 + a^2\cos^2\theta)d\theta^2 -$$
$$-(r^2 + a^2)\sin^2\theta d\varphi^2. \quad (7.205)$$

7.7.8 Solution with a variable Coulomb-Newton interaction and three-dimensional rotation of a source

(7.206)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0,1,0,0), \quad \sigma^i_{1\dot{1}} = \rho\overline{\rho}(\Omega, -Y, 0, a),$$

$$\sigma^i_{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}(ia\sin\theta, 0, 1, i\mathrm{cosec}\theta), \quad \sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}},$$

$$\sigma_i^{0\dot{0}} = (1,0,0,-a\sin^2\theta),$$

$$\sigma_i^{1\dot{1}} = \rho\overline{\rho}\left(Y, (\rho\overline{\rho})^{-1}, 0, -a\sin^2\theta Y\right),$$

$$\sigma_i^{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}\left(ia\sin\theta, 0, -(\rho\overline{\rho})^{-1}, -i\Omega\sin\theta\right), \quad \sigma_i^{1\dot{0}} = \overline{\sigma_i^{0\dot{1}}},$$

$$\Omega = r^2 + a^2, \quad Y = (r^2 + a^2 - 2\Psi^0 r)/2,$$

$$a = \mathrm{const}, \quad \Psi^0 = \Psi^0(u).$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r - ia\cos\theta)^{-1}, \beta = -\cot\theta\overline{\rho}/(2)^{3/2},$$

$$\pi = ia\sin\theta\rho^2/(2)^{1/2}, \quad \alpha = \pi - \overline{\beta}, \quad \mu = Y\rho^2\overline{\rho},$$

$$\gamma = \mu + (r + \Psi^0(u))\rho\overline{\rho}/2, \quad \tau = ia\sin\theta\rho\overline{\rho}/(2)^{1/2},$$

$$\nu = \frac{-i\dot{\Psi}^0(u)ra\sin\theta\rho^2\overline{\rho}}{2^{1/2}}.$$

4. Spinor components of the Riemannian tensor

$$\Psi_2 = \Psi = \Psi^0(u)\rho^3,$$

$$\Psi_2 = -i\dot{\Psi}^0(u)a\sin\theta\rho^2\overline{\rho}/(2)^{3/2} - 2i\dot{\Psi}^0(u)ra\sin\theta\rho^3\overline{\rho}/(2)^{1/2},$$

$$\Psi_4 = \ddot{\Psi}^0(u)ra^2\sin^2\theta\rho^3\overline{\rho}/2 + \dot{\Psi}^0(u)ra^2\sin^2\theta\rho^4\overline{\rho},$$

$$\Psi_{12} = -i\dot{\Psi}^0(u)a\sin\theta\rho^2\overline{\rho}/(2)^{3/2},$$

$$\Psi_{22} = -\ddot{\Psi}^0(u)ra^2\sin^2\theta\rho^2\overline{\rho}^2/2 - \dot{\Psi}^0(u)r^2\rho^2\overline{\rho}^2,$$

The metric of the Riemannian solution (7.206) in the coordinates (7.182) has the form

$$ds^2 = \left(1 - \frac{2\Psi^0(t)r}{r^2 + a^2\cos^2\theta}\right)c^2dt^2 + \frac{4\Psi^0(t)ra}{r^2 + a^2\cos^2\theta}\sin^2\theta d\varphi cdt -$$
$$-\frac{r^2 + a^2\cos^2\theta}{r^2 - 2\Psi^0(t)r + a^2}dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 -$$
$$-\left(r^2 + a^2 + \frac{2\Psi^0(t)ra^2}{r^2 + a^2\cos^2\theta}\sin^2\theta\right)\sin^2\theta d\varphi^2. \quad (7.207)$$

7.7.9 Solution with electronuclear interaction and three-dimensional rotation of a source

(7.208)

1. Coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi$.
2. Components of the Newman-Penrose symbols

$$\sigma^i_{0\dot{0}} = (0, 1, 0, 0), \quad \sigma^i_{1\dot{1}} = \rho\overline{\rho}(\Sigma, -\Pi, 0, a),$$

$$\sigma^i_{0\dot{1}} = -\frac{\overline{\rho}}{\sqrt{2}}(ia\sin\theta + 2ir_N\cot\theta, 0, 1, i\mathrm{cosec}\theta), \quad \sigma^i_{1\dot{0}} = \overline{\sigma^i_{0\dot{1}}},$$

$$\Sigma = r^2 + r_N^2 + a^2, \quad \Pi = (r^2 - r_N^2 + a^2 - r_e r)/2,$$

$$r_N = \mathrm{const}, \quad a = \mathrm{const}, \quad r_e = \mathrm{const}.$$

3. Spinor components of the Ricci rotation coefficients

$$\rho = -(r + ir_N - ia\cos\theta)^{-1}, \quad \beta = \overline{\rho}\beta^0, \quad \pi = \rho^2\overline{\tau}^0,$$

$$\alpha = \rho\alpha^0 + \rho^2\overline{\tau}^0, \quad \tau = \rho\overline{\rho}\tau^0,$$

$$\mu = \rho/2 + \rho\Psi^0/2 + \rho\overline{\rho}\overline{\Psi}^0/2 + \rho^2\overline{\rho}\tau^0\overline{\tau},$$

$$\gamma = \rho^2\Psi^0 + \rho\overline{\rho}(\tau^0\alpha^0 + \overline{\tau}^0\beta^0) + \rho^2\overline{\rho}\tau^0\overline{\tau}^0,$$

$$\Psi^0 = r_e/2 + ir_N,$$

$$\overline{\alpha}^0 = -\beta^0, \quad \beta^0 = -\frac{1}{4}(2)^{1/2}\cot\theta, \quad \tau^0 = -\frac{1}{2}ia(2)^{1/2}\sin\theta.$$

4. Spinor components of the Riemannian tensor

$$\Psi_2 = \Psi = \Psi^0\rho^3.$$

The nonzero components of the metric tensor g_{ij} have the form

$$\begin{aligned} g_{uu} &= \rho\overline{\rho}(r^2 r - r_N^2 + a^2\cos^2\theta), \\ g_{ur} &= 1, \\ g_{u\varphi} &= -2\rho\overline{\rho}r_N\cos\theta\Pi + 2\rho\overline{\rho}a\sin^2\theta(r_e r/2 + r_N^2), \\ g_{r\varphi} &= -a\sin^2\theta - 2r_N\cos\theta, \\ g_{\theta\theta} &= -r^2 - (r_N - a\cos\theta)^2, \\ g_{\varphi\varphi} &= \rho\overline{\rho}\Pi(a\sin^2\theta + 2r_N\cos\theta)^2 - \rho\overline{\rho}\sin^2\theta\Sigma^2. \end{aligned} \quad (7.209)$$

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